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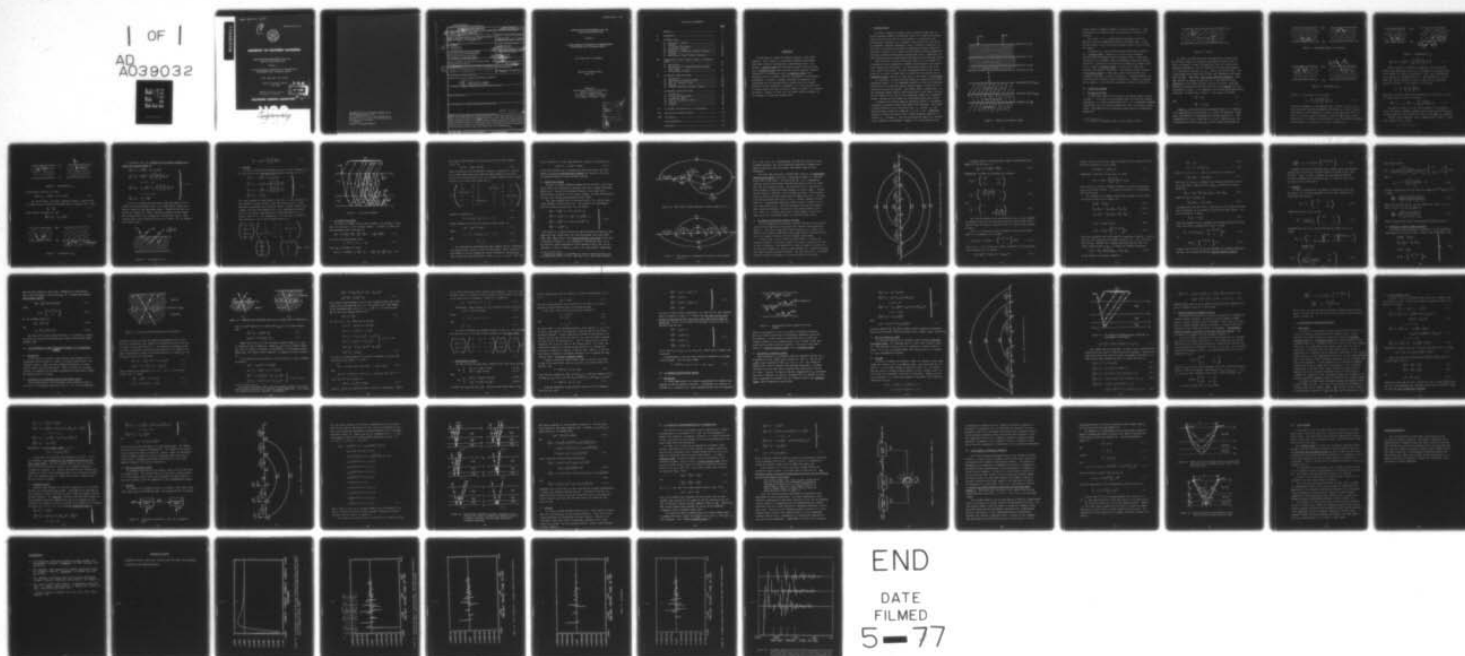
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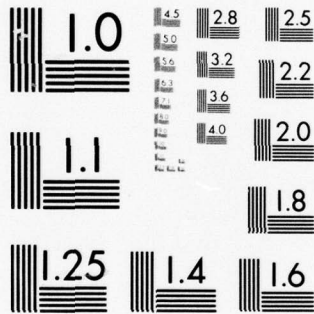
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UNIVERSITY OF SOUTHERN CALIFORNIA

INFORMATION PROCESSING FOR OIL
AND GAS EXPLORATION

Volume 3

A TIME-DOMAIN APPROACH TO SEISMOGRAM
SYNTHESIS FOR LAYERED MEDIA

N. E. Nahi and J. M. Mendel

Interim Technical Report
June 1976

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ABSTRACT

In this paper we obtain time-domain equations — state space equations — for a model of a horizontally stratified nonabsorptive earth with vertically traveling plane compressional waves. We develop a complete model which generates primaries and all multiples; a primaries model which generates just the primary reflection components of a seismogram; and a secondaries model which generates just the secondary reflection components of a seismogram. Additionally, we obtain a complete (canonical) decomposition of a seismogram into an interconnection of models each one of which generates just the primaries, secondaries, etc. We also show how to obtain transfer functions from these models by means of a recursive procedure. Finally, we show how our normal incidence models can be extended to non-normal incident waves.

I. INTRODUCTION

Ray theory models for seismic waves in layered media may be used to generate synthetic seismograms, which are useful for providing rough data that can be used for such purposes as evaluation of new deconvolution, migration, etc. techniques. These models are also useful for identifying important parameters, such as reflection coefficients. In this paper our attention is directed at obtaining time-domain equations — state space equations — for a model of a horizontally stratified nonabsorptive earth with vertically traveling plane compressional waves. Additionally, we shall demonstrate how to obtain transfer functions from the state space model.

Much work has been done in the past (Refs. 1-3, for example) in developing the basic equations for synthetic seismograms for the layered earth situation described above. We shall discuss some of this work very briefly in Section III. In order to distinguish between the earlier work and our work, we shall refer to models from the earlier work as traditional models, whereas we shall refer to our models as new models.

As a preview, we state the major differences between the new and traditional models: (1) the traditional models are derived by examining wave effects at the interface between adjacent layers, whereas the new models are derived by examining wave effects within a single layer; (2) the traditional models assume equal travel times in each layer, whereas the new models do not; and (3) the traditional models are z-transform models, whereas the new models are time-domain state space models.

A system of K layered media is depicted in Figure 1. We adopt the convention of calling the layer below layer K the basement. Each layer is characterized by its one way travel time, τ_i , velocity, V_i , and normal incidence reflection coefficient r_i ($i=1,2,\dots,K$). Additionally, interface-0 denotes the surface and is characterized by reflection coefficient r_0 . In Figure 1, $m(t)$ and $y(t)$ denote the input (e.g., seismic source signature from dynamite, aquagun, etc.) to the layered earth

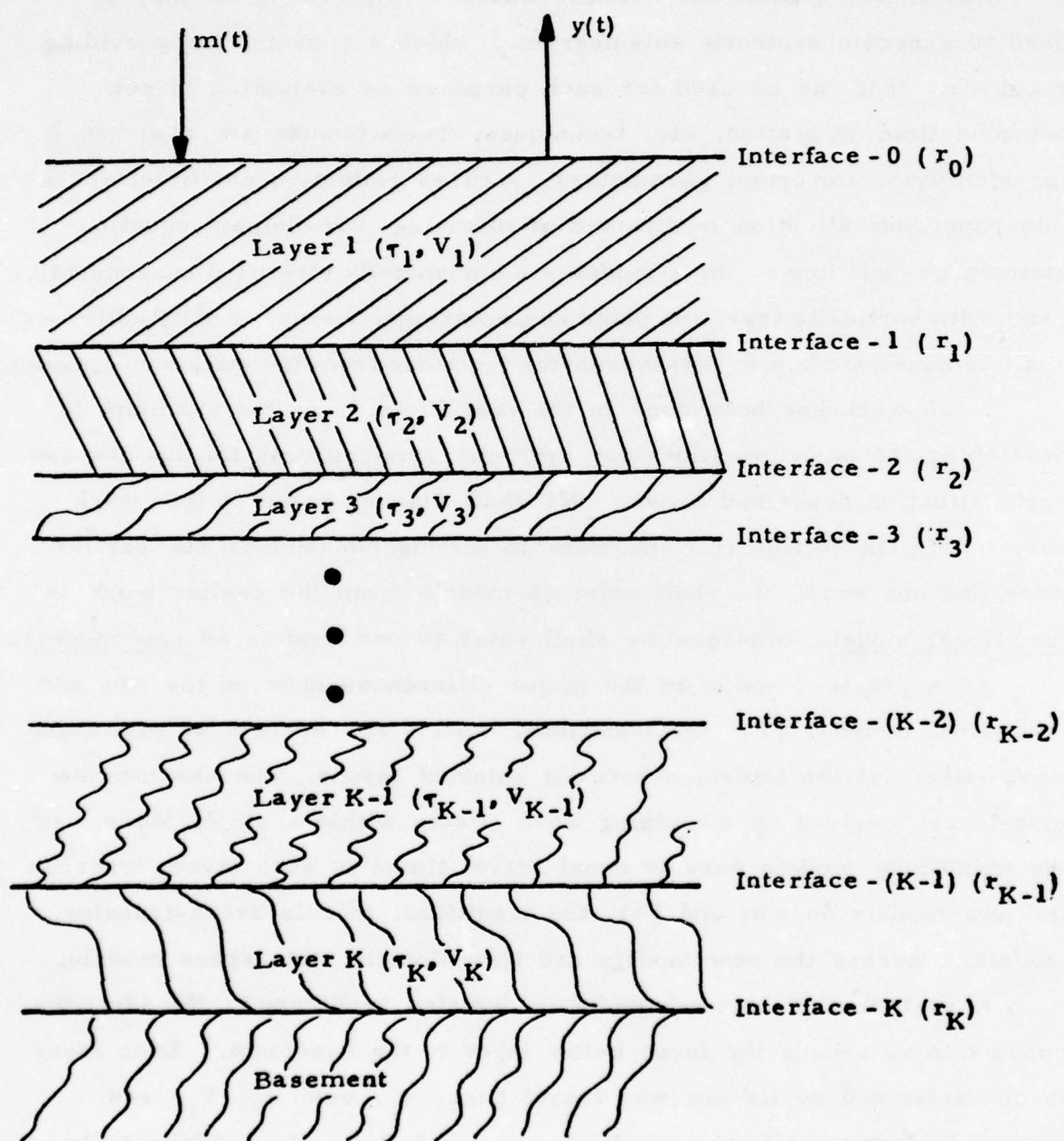


Figure 1. System of K layered media

system* which is applied at interface-0, and the output (i.e., ideal seismogram) of the system which is observed at the surface, respectively.

As in Refs. 2 & 3, we shall find it convenient to draw ray diagrams with time displacement along the horizontal axis, so that the rays appear to be at non-normal incidence and so do not overlap one another.

In Section II, we derive state equations and transfer functions for a complete model, complete in the sense that it generates primaries and all multiples for the K layered media. Connections between the new (complete) model and traditional models are made in Section III. In Section IV we show how to modify the complete model so as to obtain a primaries model — a state space model which generates only primary reflections. The extension of these results to a secondaries model — a state space model which generates only secondary reflections — is given in Section V. A generalization of the Section IV and V results is given in Section VI. Finally, we indicate how to apply our new models to non-normal incidence in Section VII.

II. COMPLETE MODEL

A. Derivation of Model

A state space model for the system of K layered media, depicted in Figure 1, is derived in this paragraph under the following modeling assumptions: (1) plane compressional waves, (2) horizontally stratified nonabsorbative layers of different thicknesses, and (3) normally incident waves. The extension to non-normally incident waves is considered in Section VII.

*

In a marine environment, layer 1 can be taken as water.

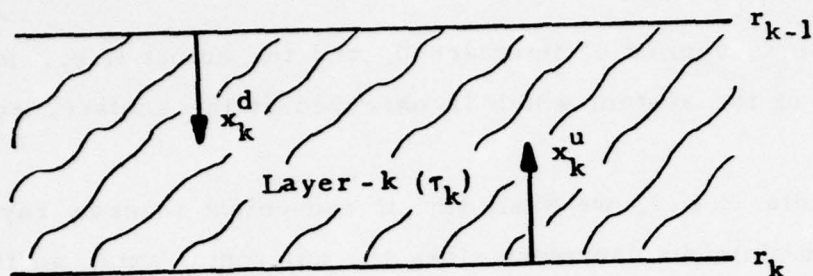


Figure 2. Layer-k

To begin, we define the states associated with a general layer, the k^{th} layer (Figure 2), keeping in mind that the logical choice of states, in general, is made to simplify the form of a model and its subsequent utilization. For the purposes of the present development, it is convenient to associate τ_k , r_{k-1} , and r_k with the k^{th} layer. The compressional waves within the k -th layer are identified by two states, $x_k^u(t)$ and $x_k^d(t)$. Physically, these states represent reflections. State $x_k^u(t)$ denotes the upward moving wave (i.e., ray) which has been reflected off of the bottom interface of layer k ; whereas, state $x_k^d(t)$ denotes the downward moving wave which has been reflected off of the top interface of layer k . To emphasize the fact that x_k^u and x_k^d represent reflections, we depict this in Figure 3, from which we see that

$$x_k^u = r_k \mu_k \quad (1)$$

and

$$x_k^d = -r_{k-1} \xi_k \quad (2)$$

Signal μ_k is the result of two phenomena (Figure 4), a reflection of an upgoing wave off of interface $k-1$ — which, by definition, is x_k^d — and a transmission of a downgoing wave through interface $k-1$. This downgoing wave is not a state (since it is not a reflection); but, can be expressed in terms of state x_{k-1}^u by the construction depicted in

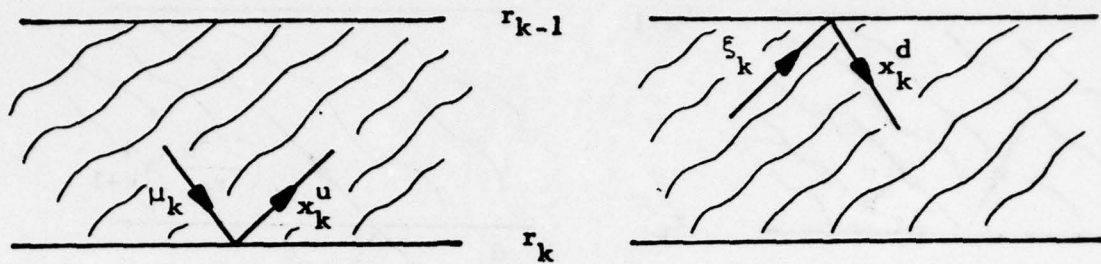


Figure 3. Interpreting states as reflections

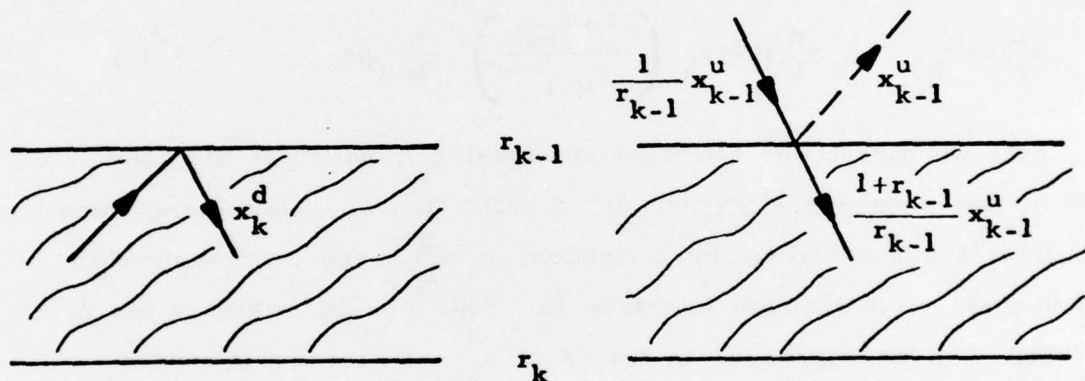


Figure 4. Components of μ_k

Figure 4. From Figure 4, we see that

$$\mu_k = x_k^d + \left(\frac{1+r_{k-1}}{r_{k-1}} \right) x_{k-1}^u \quad (3)$$

In order to complete our description of x_k^u , via Eqs. (1) and (3) we must specify the functional dependences of x_k^u and μ_k on time. Observe, from Figures 3 and 4 that Eq. (1) relates μ_k and x_k^u at the bottom of layer k , whereas μ_k has been defined in terms of x_k^d and x_{k-1}^u at the top of layer k . In order to account for the τ_k sec. time delay in traveling through layer k , we write Eqs. (1) and (3) together, as

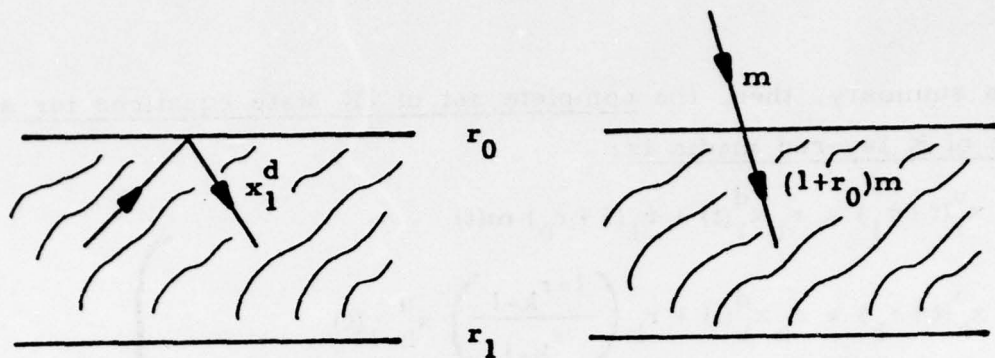


Figure 6. Components of μ_1

which means, from Eq. (1), that

$$x_1^u(t + \tau_1) = r_1 x_1^d(t) + r_1(1 + r_0) m(t) \quad (8)$$

For the K^{th} layer, we follow customary practice to assume that no signal returns up from the basement; hence, (compare Figs. 7 and 5)

$$\xi_K = x_K^u \quad (9)$$

which means, from Eq. (2), that

$$x_K^d(t + \tau_K) = -r_{K-1} x_K^u(t) \quad (10)$$

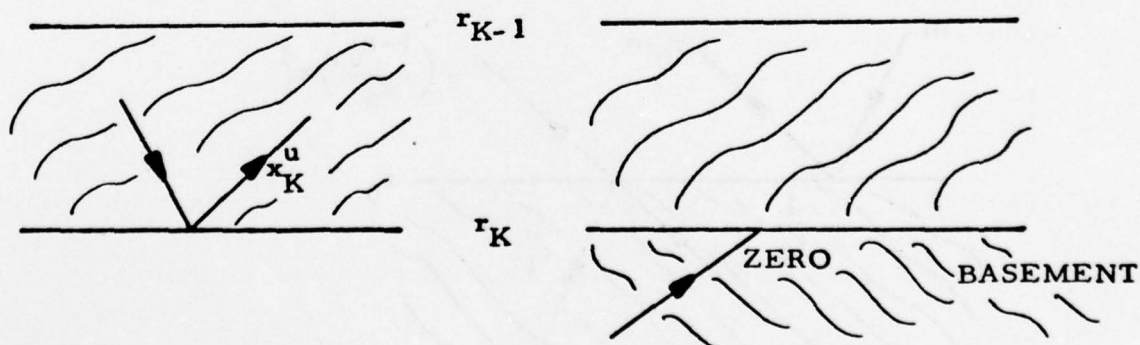


Figure 7. Components of ξ_K

In summary, then, the complete set of 2K state equations for a system of K layered media is:

$$\left. \begin{aligned}
 x_1^u(t+\tau_1) &= r_1 x_1^d(t) + r_1(1+r_0) m(t) \\
 x_k^u(t+\tau_k) &= r_k x_k^d(t) + r_k \left(\frac{1+r_{k-1}}{r_{k-1}} \right) x_{k-1}^u(t) \\
 k &= 2, 3, \dots, K \\
 x_k^d(t+\tau_k) &= -r_{k-1} x_k^u(t) + r_{k-1} \left(\frac{1-r_k}{r_k} \right) x_{k+1}^d(t) \\
 k &= 1, 2, \dots, K-1 \\
 x_K^d(t+\tau_K) &= -r_{K-1} x_K^u(t)
 \end{aligned} \right\} \quad (11)$$

In order to complete the state space model description for the system of K layered media in Fig. 1, we must express output $y(t)$ in terms of input $m(t)$ and appropriate states. Output $y(t)$ has two components (see Figure 8), a direct reflection of $m(t)$ off of the surface and a transmission of an upcoming wave through the surface. That upcoming wave can be expressed in terms of a state $-x_1^d$ — as depicted in the figure; hence,

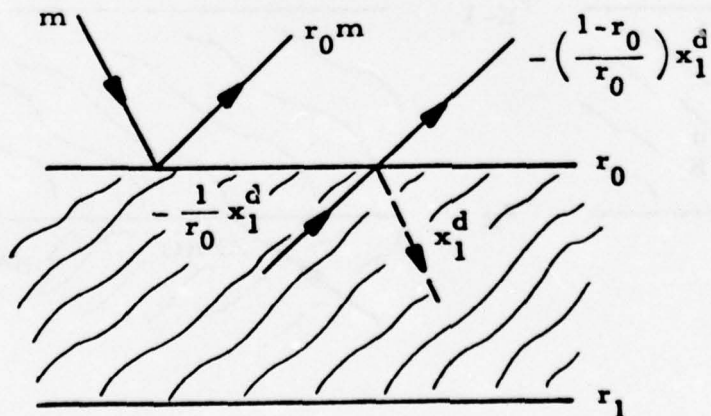


Figure 8. Components of $y(t)$

$$y(t) = r_0 m(t) - \left(\frac{1-r_0}{r_0} \right) x_1^d(t) \quad (12)$$

B. Example

In order to illustrate the detailed structure of state equations (11), we write them for the two layer media depicted in Figure 9 ($K=2$):

$$\left. \begin{aligned} x_1^u(t+\tau_1) &= r_1 x_1^d(t) + r_1(1+r_0) m(t) \\ x_1^d(t+\tau_1) &= -r_0 x_1^u(t) + r_0 \left(\frac{1-r_1}{r_1} \right) x_2^d(t) \\ x_2^u(t+\tau_2) &= r_2 x_2^d(t) + r_2 \left(\frac{1+r_1}{r_1} \right) x_1^u(t) \\ x_2^d(t+\tau_2) &= -r_1 x_2^u(t) \end{aligned} \right\} \quad (13)$$

The output equation for this system is Eq. (12). For the purposes of this example, we have chosen to order the states by layer. Other orderings are possible (see Section D). Figure 9 depicts primary and multiple reflections and illustrates the very complicated internal behavior of the system, and that $y(t)$ has components which occur at specific values of t . For compactness, it is convenient to write the four equations in (13) in vector matrix notation, as

$$\begin{pmatrix} x_1^u(t+\tau_1) \\ x_1^d(t+\tau_1) \\ x_2^u(t+\tau_2) \\ x_2^d(t+\tau_2) \end{pmatrix} = \begin{pmatrix} 0 & r_1 & 0 & 0 \\ -r_0 & 0 & 0 & r_0(1-r_1)/r_1 \\ r_2(1+r_1)/r_1 & 0 & 0 & r_2 \\ 0 & 0 & -r_1 & 0 \end{pmatrix} \times \begin{pmatrix} x_1^u(t) \\ x_1^d(t) \\ x_2^u(t) \\ x_2^d(t) \end{pmatrix} + \begin{pmatrix} r_1(1+r_0) \\ 0 \\ 0 \\ 0 \end{pmatrix} m(t) \quad (14)$$

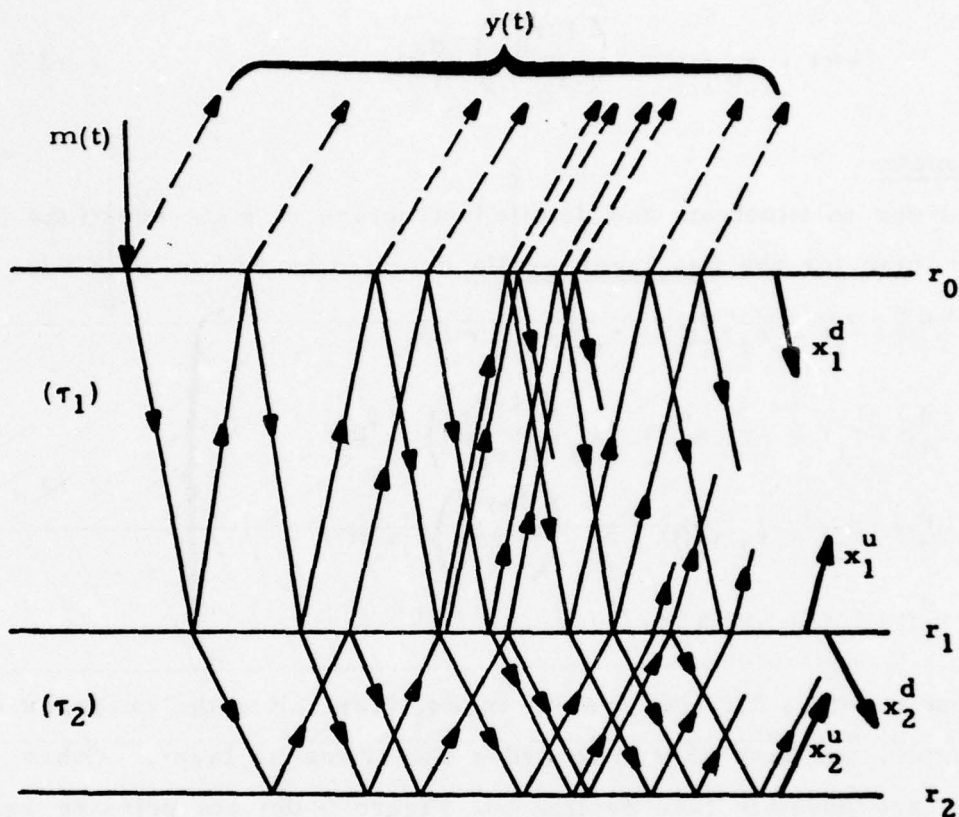


Figure 9. Two Layer Example

C. A Compact Description

By means of suitable vector matrix notation, it is possible to write Eqs. (11) and (12) in a very compact manner. To begin, we define the $2K$ -dimensional state vector $\underline{x}(t)$, as

$$\underline{x}(t) = \text{col} [x_1^u(t), x_1^d(t), x_2^u(t), x_2^d(t), \dots, x_K^u(t), x_K^d(t)] \quad (15)$$

Let $\underline{\tau}$ be the K -dimensional vector

$$\underline{\tau} = \text{col} (\tau_1, \tau_2, \dots, \tau_K) \quad (16)$$

Then, $\underline{x}(t + \underline{\tau})$ denotes the vector

$$\underline{x}(t + \underline{\tau}) = \text{col} [x_1^u(t + \tau_1), x_1^d(t + \tau_1), \dots, x_K^u(t + \tau_K), x_K^d(t + \tau_K)] \quad (17)$$

By means of this notation, we can now write the $2K$ state equations in (11), as

$$\underline{x}(t+\tau) = A \underline{x}(t) + B m(t) \quad (18)$$

where A and B are $2K \times 2K$ and $2K \times 1$ matrices, respectively. Matrix A is a band matrix with zeros along its main diagonal; its structure for $K=2$ is evident from Eq. (14). As a further example of its structure we give A for a three layer media ($K=3$):

$$A = \begin{pmatrix} 0 & r_1 & 0 & 0 & 0 & 0 \\ -r_0 & 0 & 0 & r_0(1-r_1)/r_1 & 0 & 0 \\ r_2(1+r_1)/r_1 & 0 & 0 & r_2 & 0 & 0 \\ 0 & 0 & -r_1 & 0 & 0 & r_1(1-r_2)/r_2 \\ 0 & 0 & r_3(1+r_2)/r_2 & 0 & 0 & r_3 \\ 0 & 0 & 0 & 0 & -r_2 & 0 \end{pmatrix} \quad (19)$$

Matrix B is defined, as

$$B = \text{col} [r_1(1+r_0), 0, 0, \dots, 0] \quad (20)$$

Output equation (12) can now be written in terms of $\underline{x}(t)$, as

$$y(t) = C \underline{x}(t) + D m(t) \quad (21)$$

where

$$C = (0, -(1-r_0)/r_0, 0, \dots, 0) \quad (22)$$

and

$$D = r_0 \quad (23)$$

It is important to understand that state equation (18) is a continuous-time equation that involves a vector of time delays. Some remarks on the solution of such an equation are given in Section D. In the special case of equal one way travel times, where $\tau_1 = \tau_2 = \dots = \tau_K \triangleq \tau$, Eq. (18)

can be reduced to a vector finite-difference equation by choosing $t = k\tau$:

$$\underline{x}[(k+1)\tau] = A \underline{x}(k\tau) + B m(k\tau) \quad (24)$$

In the more general case of nonequal one way travel times, Eq. (18) applies; it is not a finite-difference equation, but belongs to the class of equations known as causal functional equations.

D. Signal Flow Graphs

In order to develop additional insight into the structure of our state space model, it is instructive to portray that model in signal flow graph (SFG) form.* Because all important features of our model can be observed from the nature of a two layer example, we shall develop SFG's for that case, leaving generalizations to the reader.

The state equations for $K=2$ are given in Eq. (13), and the observation equation is given in Eq. (12). For purposes of constructing a SFG it is useful to rewrite Eq. (13), as follows:

$$\left. \begin{aligned} x_1^u(t) &= r_1 x_1^d(t - \tau_1) + r_1(1 + r_0) m(t - \tau_1) \\ x_1^d(t) &= -r_0 x_1^u(t - \tau_1) + r_0[(1 - r_1)/r_1] x_2^d(t - \tau_1) \\ x_2^u(t) &= r_2 x_2^d(t - \tau_2) + r_2[(1 + r_1)/r_1] x_1^u(t - \tau_2) \\ x_2^d(t) &= -r_1 x_2^u(t - \tau_2) \end{aligned} \right\} \quad (25)$$

A SFG (which is an interconnection of nodes and directed branches, with input node $m(t)$, output node; $y(t)$, and intermediate state nodes $x_1^u(t)$, $x_1^d(t)$, $x_2^u(t)$, $x_2^d(t)$) based on the layered-ordering of the states in Eq. (25) is depicted in Figure 10. For simplicity in drawing the SFG, we do not show the explicit dependence of node signals on time; and, we use operators z_1 and z_2 to denote τ_1 and τ_2 sec. time delays, respectively

* Signal flow graphs are essentially the same as block diagrams, but are artistically easier to draw. See Ref. 4 for a discussion on SFG's.

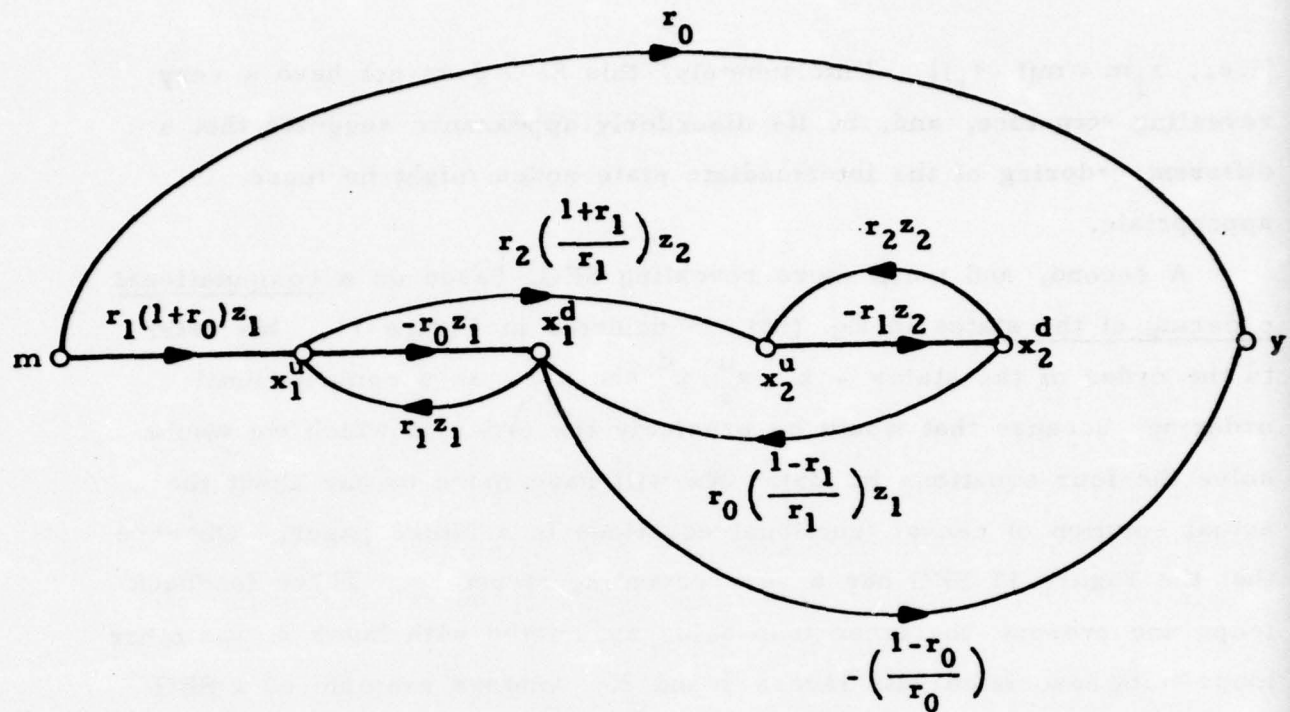


Figure 10. SFG based on layer-ordering of nodes (states) for $K=2$.

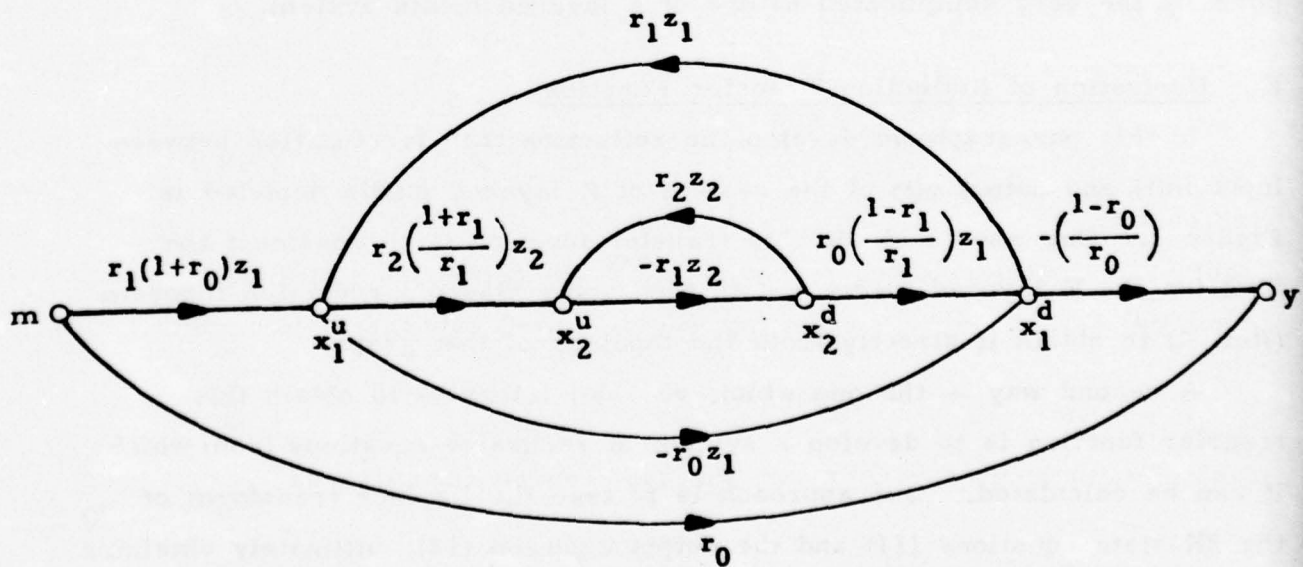


Figure 11. SFG based on computational ordering of nodes (states) for $K=2$.

[i.e., $z_1 m = m(t - \tau_1)$]. Unfortunately, this SFG does not have a very revealing structure, and, by its disorderly appearance suggests that a different ordering of the intermediate state nodes might be more appropriate.

A second, and much more revealing SFG, based on a computational ordering of the states in Eq. (25), is depicted in Figure 11. We refer to the order of the states - x_1^u, x_2^u, x_2^d and x_1^d - as a computational ordering, because that would be precisely the order in which we would solve the four equations in (25). We will have more to say about the actual solution of causal functional equations in a future paper. Observe that the Figure 11 SFG has a very revealing structure. Three feedback loops are present; the inner loop being associated with layer 2, the other loops being associated with layers 1 and 2. Another example of a SFG based on a computational ordering of states is given in Figure 12 for a three layer media ($K=3$). This SFG illustrates not only the feedback paths caused by layerings, but also the many forward paths as well. (i.e., there are 6 feedback paths and 4 forward paths), and serves to point up the very complicated nature of a layered media system.

E. Derivation of Reflection Transfer Function

In this paragraph we develop the reflection transfer function between input $m(t)$ and output $y(t)$ of the system of K layered media depicted in Figure 1. One way to obtain this transfer function is to construct the SFG for the K layered media and to then apply Mason's reduction theorem (Ref. 4) to obtain it directly from the topology of that graph.

A second way - the one which we shall follow - to obtain this transfer function is to develop a system of recursive equations from which it can be calculated. Our approach is to take the Laplace transform of the $2K$ state equations (11) and the output equation (12), ultimately obtaining the Laplace transfer function $Y(s)/M(s)$. We shall proceed in a systematic manner focusing our attention at a layered-ordering of the states.

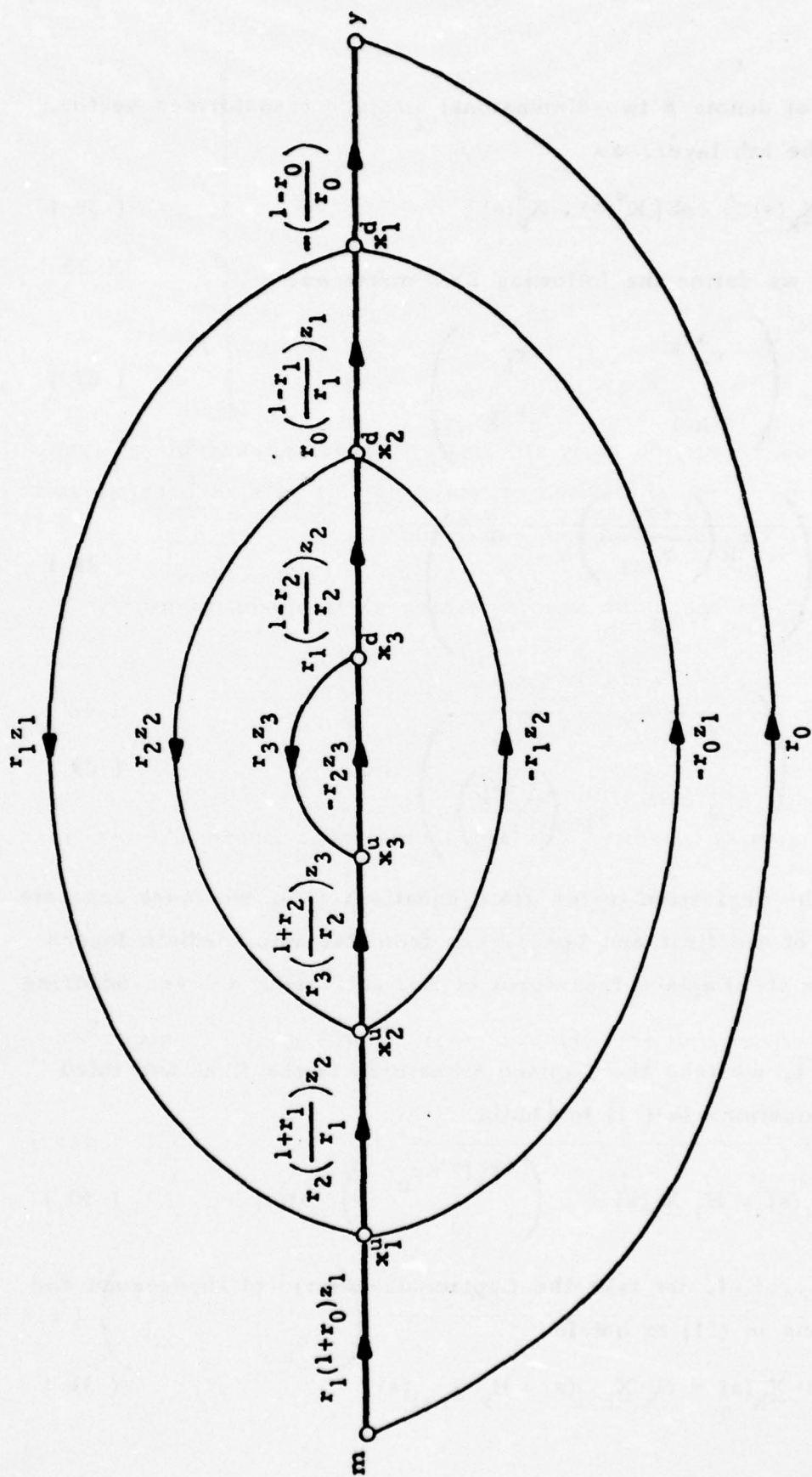


Figure 12. SFG based on computational ordering of nodes (states) for $K=3$.

Let $\underline{X}_k(s)$ denote a two-dimensional Laplace transformed vector, defined for the k th layer, as

$$\underline{X}_k(s) \triangleq \text{col} [X_k^u(s), X_k^d(s)] \quad (26)$$

Additionally, we define the following 2×2 matrices:

$$F_k(s) = \begin{pmatrix} e^{s\tau_k} & -r_k \\ r_{k-1} & e^{s\tau_k} \end{pmatrix} \quad (27)$$

$$G_k = \begin{pmatrix} r_k \left(\frac{1+r_{k-1}}{r_{k-1}} \right) & 0 \\ 0 & 0 \end{pmatrix} \quad (28)$$

and

$$H_k = \begin{pmatrix} 0 & 0 \\ 0 & r_{k-1} \left(\frac{1-r_k}{r_k} \right) \end{pmatrix} \quad (29)$$

As in the derivation of the state equations (11), we must separate the analysis of the first and last layers from the intermediate layers when we take the Laplace transform of Eq. (11) using a layer-ordering of states.

For $k=1$, we take the Laplace transform of the first and third (with $k=1$) equations in (11) to obtain

$$F_1(s) \underline{X}_1(s) = H_1 \underline{X}_2(s) + \begin{pmatrix} r_1(1+r_0) \\ 0 \end{pmatrix} M(s) \quad (30)$$

For $k=2, 3, \dots, K-1$, we take the Laplace transform of the second and third equations in (11) to obtain

$$F_k(s) \underline{X}_k(s) = G_k \underline{X}_{k-1}(s) + H_k \underline{X}_{k+1}(s) \quad (31)$$

Finally, for $k=K$ we take the Laplace transform of the second (for $k=K$) and fourth equations in (11) to obtain

$$F_K(s) \underline{X}_K(s) = G_K \underline{X}_{K-1}(s) \quad (32)$$

Additionally, from Eqs. (12) and (26), we obtain

$$Y(s) = r_0 M(s) - \left(\frac{1-r_0}{r_0} \right) (0 \ 1) \underline{X}_1(s) \quad (33)$$

from which we see that to compute $Y(s)/M(s)$ we must first determine $\underline{X}_1(s)$ as a function of $M(s)$. A recursive procedure for accomplishing this is given next.

The complete set of Laplace transformed equations for the system of K layered media can be written, as:

$$F_K \underline{X}_K = G_K \underline{X}_{K-1} \quad (34a)$$

$$F_{K-1} \underline{X}_{K-1} = G_{K-1} \underline{X}_{K-2} + H_{K-1} \underline{X}_K \quad (34b)$$

$$F_{K-2} \underline{X}_{K-2} = G_{K-2} \underline{X}_{K-3} + H_{K-2} \underline{X}_{K-1} \quad (34c)$$

. . .

$$F_2 \underline{X}_2 = G_2 \underline{X}_1 + H_2 \underline{X}_3 \quad (34d)$$

$$F_1 \underline{X}_1 = H_1 \underline{X}_2 + \begin{pmatrix} r_1(1+r_0) \\ 0 \end{pmatrix} M \quad (34e)$$

where the argument s has been omitted for notational simplicity. Observe, from Eq. (27), that $F_k(s)$ is invertible for all $k=1,2,\dots,K$. Solve Eq. (34a) for \underline{X}_K , substitute the resulting expression into Eq. (34b), and collect common terms in \underline{X}_{K-1} , to obtain the following expression:

$$(F_{K-1} - H_{K-1} F_K^{-1} G_K) \underline{X}_{K-1} = G_{K-1} \underline{X}_{K-2} \quad (35)$$

Let W_j denote a 2×2 matrix, defined as

$$W_K = F_K \quad (36a)$$

$$W_j = F_j - H_j W_{j+1}^{-1} G_{j+1} \quad (36b)$$

where $j = K-1, K-2, \dots, 1$. Then, Eq. (35) can be rewritten as

$$W_{K-1} \underline{X}_{K-1} = G_{K-1} \underline{X}_{K-2} \quad (37)$$

Solve Eq. (37) for \underline{X}_{K-1} , substitute the resulting expression into Eq. (34c), and collect common terms in \underline{X}_{K-2} , to obtain the following expression:

$$(F_{K-2} - H_{K-2} W_{K-1}^{-1} G_{K-1}) \underline{X}_{K-2} = G_{K-2} \underline{X}_{K-3} \quad (38)$$

which can also be written, as

$$W_{K-2} \underline{X}_{K-2} = G_{K-2} \underline{X}_{K-3} \quad (39)$$

Comparing Eqs. (39) and (37) it is clear that we can continue this backward development of our solution for $\underline{X}_1(s)$ in the same manner. For the second layer, we obtain

$$W_2 \underline{X}_2 = G_2 \underline{X}_1 \quad (40)$$

Solve this equation for \underline{X}_2 , substitute the resulting expression into Eq. (34e), and collect common terms in \underline{X}_1 , to obtain

$$(F_1 - H_1 W_2^{-1} G_2) \underline{X}_1 = \begin{pmatrix} r_1(1+r_0) \\ 0 \end{pmatrix} M \quad (41)$$

or,

$$W_1 \underline{X}_1 = \begin{pmatrix} r_1(1+r_0) \\ 0 \end{pmatrix} M \quad (42)$$

Finally, solve Eq. (42) for \underline{X}_1 and substitute the resulting expression into Eq. (33) to obtain the desired reflection transfer function

$$\frac{Y(s)}{M(s)} = r_0 - (0 \ 1) W_1^{-1}(s) \begin{pmatrix} r_1(1-r_0^2)/r_0 \\ 0 \end{pmatrix} \quad (43)$$

Observe that $Y(s)/M(s)$ depends only on matrix $W_1(s)$, and that that matrix must be solved in a recursive manner by means of Eqs. (36a) and (36b). These recursive equations are quite different from the recursive equations given by Robinson (Ref. 3), which are only valid for equal one way travel times.

F. Example

In order to illustrate the calculation of $Y(s)/M(s)$ we carry out the steps in our recursive algorithm for a two layer example ($K=2$). From Eqs. (36a) and (27),

$$W_2 = F_2 = \begin{pmatrix} e^{s\tau_2} & -r_2 \\ r_1 & e^{s\tau_2} \end{pmatrix} \quad (44)$$

Taking the inverse of W_2 , we find

$$W_2^{-1} = \frac{1}{e^{2\tau_2 s} + r_1 r_2} \begin{pmatrix} e^{s\tau_2} & r_2 \\ -r_1 & e^{s\tau_2} \end{pmatrix} \quad (45)$$

Substitute Eqs. (45), (27), (28) and (29) into Eq. (36b), for $j=1$, to obtain

$$W_1 = F_1 - \begin{pmatrix} 0 & 0 \\ 0 & r_0(1-r_1)/r_1 \end{pmatrix} \begin{pmatrix} e^{s\tau_2} & r_2 \\ -r_1 & e^{s\tau_2} \end{pmatrix} \begin{pmatrix} r_2(1+r_1)/r_1 & 0 \\ 0 & 0 \end{pmatrix} \\ \frac{1}{e^{2\tau_2 s} + r_1 r_2} \\ W_1 = \begin{pmatrix} e^{s\tau_1} & -r_1 \\ \frac{r_0 e^{2\tau_2 s} + r_0 r_2 / r_1}{e^{2\tau_2 s} + r_1 r_2} & e^{s\tau_1} \end{pmatrix} \quad (46)$$

from which we find

$$W_1^{-1} = \frac{e^{2\tau_2 s} + r_1 r_2}{e^{2(\tau_1 + \tau_2)s} + r_1 r_2 e^{2\tau_1 s} + r_1 r_0 e^{2\tau_2 s} + r_0 r_2} \begin{pmatrix} e^{s\tau_1} & r_1 \\ -w_1(2,1) & e^{s\tau_1} \end{pmatrix} \quad (47a)$$

where

$$w_1(2,1) = \frac{r_0 e^{2\tau_2 s} + r_0 r_2 / r_1}{e^{s\tau_1} + r_1 r_2} e^{s\tau_1} \quad (47b)$$

Finally, substitute Eq. (47) into Eq. (43), letting $\bar{z}_1 = e^{2\tau_1 s}$ and $\bar{z}_2 = e^{2\tau_2 s}$, to obtain

$$\frac{Y(s)}{M(s)} = \frac{r_0 \bar{z}_1 \bar{z}_2 + r_0 r_1 r_2 \bar{z}_1 + r_1 \bar{z}_2 + r_2}{\bar{z}_1 \bar{z}_2 + r_1 r_2 \bar{z}_1 + r_1 r_0 \bar{z}_2 + r_0 r_2} \quad (48)$$

which is the desired result. For the special case where $\tau_1 = \tau_2$, so that $\bar{z}_1 = \bar{z}_2 \triangleq \bar{z}$, Eq. (48) simplifies to

$$\frac{Y(s)}{M(s)} = \frac{r_0 \bar{z}^2 + (r_1 + r_0 r_1 r_2) \bar{z} + r_2}{\bar{z}^2 + (r_1 r_2 + r_1 r_0) \bar{z} + r_0 r_2} \quad (49)$$

which is precisely the same result derived by Robinson in Ref. 3.

G. Derivation of State Transfer Functions

By means of our derivation of $Y(s)/M(s)$, it is also possible to obtain $\underline{X}_k(s)$ as a function of $M(s)$. Recall, from Paragraph E, that

$$\left. \begin{aligned} W_K \underline{X}_K &= G_K \underline{X}_{K-1} \\ W_{K-1} \underline{X}_{K-1} &= G_{K-1} \underline{X}_{K-2} \\ &\vdots \\ W_2 \underline{X}_2 &= G_2 \underline{X}_1 \\ W_1 \underline{X}_1 &= \begin{pmatrix} r_1(1+r_0) \\ 0 \end{pmatrix} M \end{aligned} \right\} \quad (50)$$

Solve the last equation in (50) for \underline{X}_1 , substitute the result into the next to the last equation, solve it for \underline{X}_2 , etc. to obtain the following state transfer function

$$\underline{X}_K(s) = \prod_{i=k}^1 [W_i^{-1}(s) G_i] M(s) \quad (51)$$

where

$$G_1 \triangleq \begin{pmatrix} r_1(1+r_0) \\ 0 \end{pmatrix} \quad (52)$$

So, for example, when $K=2$,

$$\underline{X}_1 = W_1^{-1} G_1 M \quad (53a)$$

and

$$\underline{X}_2 = W_2^{-1} G_2 W_1^{-1} G_1 M \quad (53b)$$

By means of the state transfer functions, it is possible to compute the states of a K layered media directly in terms of the seismic input signature $m(t)$.

III. RELATIONSHIP OF NEW COMPLETE MODEL TO TRADITIONAL MODEL

A. Introduction

In this section we develop a state equation representation for the traditionally-derived model of a system of K layered media (Ref. 3, for example); however, unlike the more traditional derivations, we shall assume unequal one way travel times. Then, by means of a two-layer example, we establish the coordinate transformation matrix which relates the traditional and new models.

B. Derivation of a Traditionally-Derived State Space Model

The starting point for the development of the traditional model is the Figure 13 ray diagram (taken from Ref. 3). We use the (traditional)

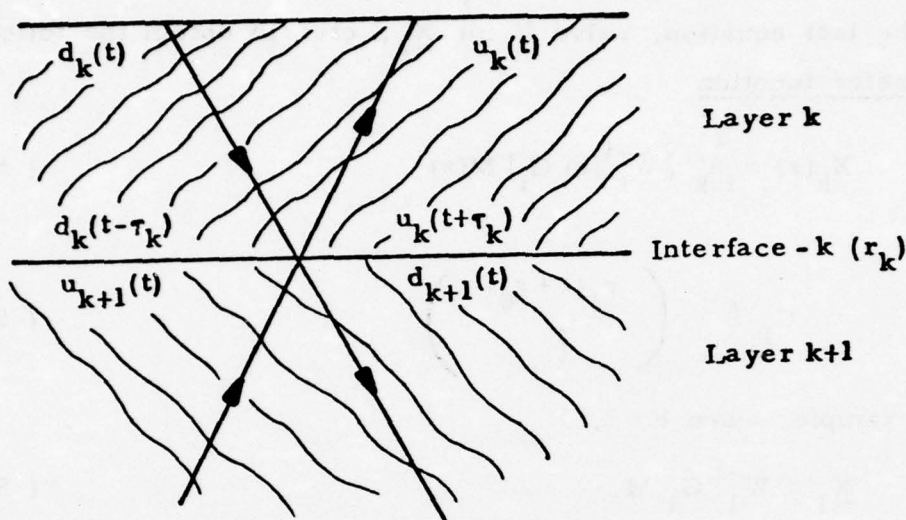


Figure 13. Reflected and transmitted waves at interface k

symbols u_k and d_k to denote the upgoing and downgoing waves in the k th layer, respectively; and, adopt the convention that waves at the top of a layer occur at present time, t . To develop the traditional model we direct our attention at the intersection point of the ray diagram and apply superposition to obtain the following equations for signals $u_k(t + \tau_k)$ and $d_{k+1}(t)$, which leave that point:

$$u_k(t + \tau_k) = r_k d_k(t - \tau_k) + (1 - r_k) u_{k+1}(t) \quad (1)$$

$$d_{k+1}(t) = (1 + r_k) d_k(t - \tau_k) - r_k u_{k+1}(t) \quad (2)$$

These equations are applicable for $k = 1, 2, \dots, K-1$. At the surface (Figure 14a), we obtain

$$u_0(t) = r_0 d_0(t) + (1 - r_0) u_1(t) \quad (3)$$

$$d_1(t) = (1 + r_0) d_0(t) - r_0 u_1(t) \quad ; \quad (4)$$

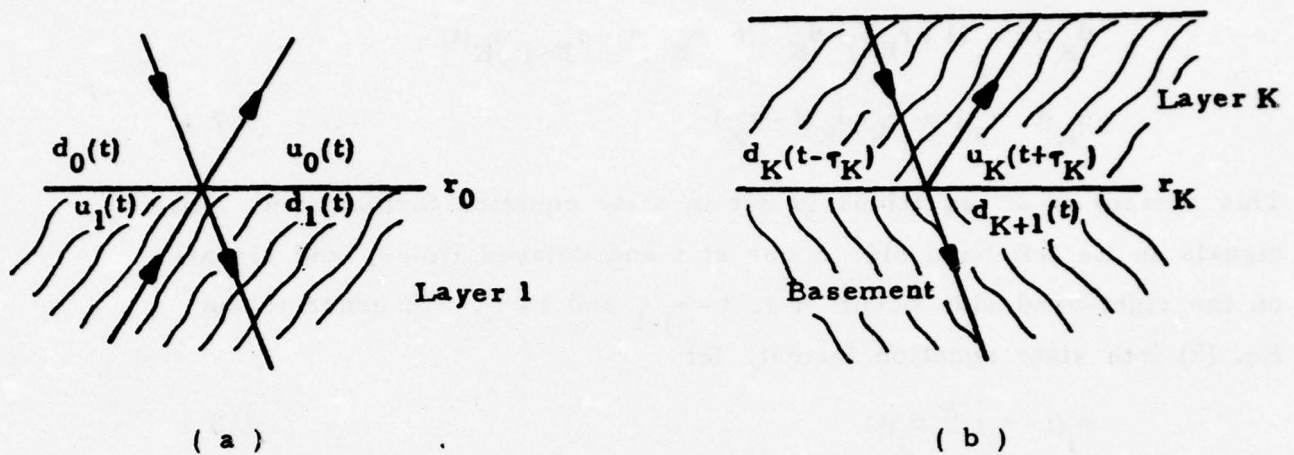


Figure 14. Reflected and transmitted waves at (a) surface and (b) interface K.

and, at the K^{th} interface, we assume that $u_{K+1}(t) = 0$, to obtain (Figure 14b)

$$u_K(t + \tau_K) = r_K d_K(t - \tau_K) \quad (5)$$

$$d_{K+1}(t) = (1 + r_K) d_K(t - \tau_K) \quad (6)$$

Signal $u_0(t)$ in Eq. (3) is the measurable system output; i.e., $u_0(t)$ is analogous to $y(t)$ in Section II. Signal $d_{K+1}(t)$ is also a system output; but, since it cannot be measured, we shall ignore it in following analyses. Signal $d_0(t)$, the seismic source signature, is analogous to $m(t)$ in Section II.

It is convenient to group Eqs. (1), (2), (4) and (5) in a layer ordering, as follows*:

$$d_1(t) = -r_0 u_1(t) + (1 + r_0) d_0(t)$$

$$u_1(t + \tau_1) = r_1 d_1(t - \tau_1) + (1 - r_1) u_2(t)$$

$$d_j(t) = (1 + r_{j-1}) d_{j-1}(t - \tau_{j-1}) - r_{j-1} u_j(t)$$

$$u_j(t + \tau_j) = r_j d_j(t - \tau_j) + (1 - r_j) u_{j+1}(t)$$

} \quad j = 2, 3, \dots, K-1

* The reader may wonder why we list the downgoing signal in each layer first, whereas in Section II we listed the upgoing signal first. The reason for doing this will become apparent in Paragraph C.

$$d_K(t) = (1+r_{K-1}) d_{K-1}(t-\tau_{K-1}) - r_{K-1} u_K(t)$$

$$u_K(t+\tau_K) = r_K d_K(t-\tau_K) \quad (7)$$

This system of $2K$ equations is not in state equation format, yet, since signals in its left-hand side occur at t and delayed times, and signals on the right-hand side occur at t , $t-\tau_{j-1}$ and $t-\tau_j$. In order to put Eq. (7) into state equation format, let

$$e_j(t+\tau_j) \triangleq d_j(t) \quad (8)$$

for all $j=1, 2, \dots, K$. Then, Eq. (7) becomes

$$e_1(t+\tau_1) = -r_0 u_1(t) + (1+r_0) d_0(t)$$

$$u_1(t+\tau_1) = r_1 e_1(t) + (1-r_1) u_2(t)$$

$$\left. \begin{aligned} e_j(t+\tau_j) &= (1+r_{j-1}) e_{j-1}(t) - r_{j-1} u_j(t) \\ u_j(t+\tau_j) &= r_j e_j(t) + (1-r_j) u_{j+1}(t) \end{aligned} \right\} \quad j=2, 3, \dots, K-1$$

$$e_K(t+\tau_K) = (1+r_{K-1}) e_{K-1}(t) - r_{K-1} u_K(t)$$

$$u_K(t+\tau_K) = r_K e_K(t) \quad (9)$$

By means of transformation (8) each pair of equations in (9) now only involves two time points, $t+\tau_j$ and t .

Let

$$\underline{x}(t) = \text{col} [e_1(t), u_1(t), e_2(t), u_2(t), \dots, e_K(t), u_K(t)] \quad (10)$$

and

$$\underline{x}(t+\underline{\tau}) = \text{col} [e_1(t+\tau_1), u_1(t+\tau_1), \dots, e_K(t+\tau_K), u_K(t+\tau_K)] \quad (11)$$

Then, Eq. (9) can be written as

$$\underline{x}(t+\underline{\tau}) = A_{\underline{x}} \underline{x}(t) + B_{\underline{x}} d_0(t) \quad (12)$$

where $A_{\underline{x}}$ and $B_{\underline{x}}$ are $2K \times 2K$ and $2K \times 1$ matrices, respectively. Matrix

A_χ is a band matrix with zeros along its main diagonal. It has zero and non-zero entries in exactly the same locations as does matrix A [see Eqs. (II-14) and (II-19) for examples]. Matrix B_χ is defined as

$$B_\chi = \text{col} [1+r_0, 0, 0, \dots, 0] \quad (13)$$

Additionally, output equation (3) can be written in terms of $\underline{x}(t)$, as

$$u_0(t) = C_\chi \underline{x}(t) + D_\chi d_0(t) \quad (14)$$

where

$$C_\chi = (0, 1-r_0, 0, \dots, 0) \quad (15)$$

and

$$D_\chi = r_0 \quad (16)$$

In order to illustrate the detailed structure of Eq. (12), and especially the structure of A_χ , we write out that equation for a two layer media:

$$\begin{pmatrix} e_1(t+\tau_1) \\ u_1(t+\tau_1) \\ e_2(t+\tau_2) \\ u_2(t+\tau_2) \end{pmatrix} = \begin{pmatrix} 0 & -r_0 & 0 & 0 \\ r_1 & 0 & 0 & 1-r_1 \\ 1+r_1 & 0 & 0 & -r_1 \\ 0 & 0 & r_2 & 0 \end{pmatrix} \begin{pmatrix} e_1(t) \\ u_1(t) \\ e_2(t) \\ u_2(t) \end{pmatrix} + \begin{pmatrix} 1+r_0 \\ 0 \\ 0 \\ 0 \end{pmatrix} d_0(t) \quad (17)$$

C. Relationship of Models

We have derived two state space models for the same linear system:

$$S_N \quad \left\{ \begin{array}{l} \underline{x}(t+\tau) = A \underline{x}(t) + B m(t) \\ y(t) = C \underline{x}(t) + D m(t) \end{array} \right. \quad \begin{array}{l} (II-18) \\ (II-21) \end{array}$$

and

$$S_T \quad \left\{ \begin{array}{l} \underline{x}(t+\tau) = A_\chi \underline{x}(t) + B_\chi d_0(t) \\ u_0(t) = C_\chi \underline{x}(t) + D_\chi d_0(t) \end{array} \right. \quad \begin{array}{l} (III-12) \\ (III-14) \end{array}$$

in which $m(t) = d_0(t)$ and $y(t) = u_0(t)$. Because of the linear nature of S_N

and S_T , $\underline{x}(t)$ and $\underline{\chi}(t)$ must be related by a linear transformation, say T , i.e.,

$$\underline{x}(t) = T \underline{\chi}(t) \quad (18)$$

We wish to determine the exact structure of T so that we can make a physical connection between state vectors $\underline{x}(t)$ and $\underline{\chi}(t)$.

If Eq. (18) is true, then S_N and S_T are equivalent if

$$A = T A_{\chi} T^{-1} \quad (19)$$

$$B = T B_{\chi} \quad (20)$$

and

$$C = C_{\chi} T^{-1} \quad (21)$$

We have before us the following problem: given matrices A , A_{χ} , B , B_{χ} , C and C_{χ} , find matrix T such that Eqs. (19), (20) and (21) are simultaneously satisfied. A direct approach to solution of this problem is to rewrite (19) and (20) as $AT = TA_{\chi}$ and $CT = C_{\chi}$, and solve these equations plus Eq. (20) for the elements of T . This is a large system of linear equations; but, it can be solved in this direct manner. A second, and much easier approach to solution of this problem is to compare the structures of matrices A and A_{χ} , B and B_{χ} , and C and C_{χ} , to observe that nonzero elements in comparable matrices occur in exactly the same location; hence, T must be a diagonal matrix.

Consider a two layer media system, where A , B and C are given in Eqs. (II-14) and (II-12), and, A_{χ} , B_{χ} and C_{χ} are given in Eqs. (17) and (15). Let

$$T = \text{diag} (t_{11}, t_{22}, t_{33}, t_{44}) \quad ; \quad (22)$$

then, from the equation $B = TB_{\chi}$, we find $t_{11} = r_1$; from the equation $CT = C_{\chi}$, we find $t_{22} = -r_0$; and, from $AT = TA_{\chi}$ we find $t_{33} = r_2$ and $t_{44} = -r_1$; hence,

$$T = \text{diag} (r_1, -r_0, r_2, -r_1) \quad (23)$$

Using the definitions of $\underline{x}(t)$ and $\underline{\chi}(t)$, and, Eq. (23), we establish, from Eq. (18), that

$$\left. \begin{aligned}
 x_1^u(t) &= r_1 e_1(t) = r_1 d_1(t - \tau_1) \\
 x_1^d(t) &= -r_0 u_1(t) \\
 x_2^u(t) &= r_2 e_2(t) = r_2 d_2(t - \tau_2) \\
 x_2^d(t) &= -r_1 u_2(t)
 \end{aligned} \right\} \quad (24)$$

By means of these simple relationships, we relate the new and traditional states as depicted in Figure 15. Observe that the states in the model derived in Section II are the reflections of the states in the model of Section III. In fact, if we compare the following versions of Eqs. (II-1) and (II-2) with Eq. (24),

$$\left. \begin{aligned}
 x_1^u(t) &= r_1 \mu_1(t - \tau_1) \\
 x_1^d(t) &= -r_0 \xi_1(t - \tau_1) \\
 x_2^u(t) &= r_2 \mu_2(t - \tau_2) \\
 x_2^d(t) &= -r_1 \xi_1(t - \tau_2)
 \end{aligned} \right\} \quad (25)$$

we see that $d_1 = \mu_1$, $d_2 = \mu_2$, $u_1 = \xi_1$ and $u_2 = \xi_2$, which further supports this conclusion.

Generalizations of the preceding results are obtained in a straightforward manner. For a K layer media,

$$T = \text{diag}(r_1, -r_0, r_2, -r_1, \dots, r_K, -r_{K-1}) \quad (26)$$

IV. A PRIMARY REFLECTION MODEL

A. Introduction

In certain applications it is useful to approximate the complete seismogram by just the primary reflections. Conceptually, the primary reflections (i.e., primaries) are easily obtained directly from ray diagrams;

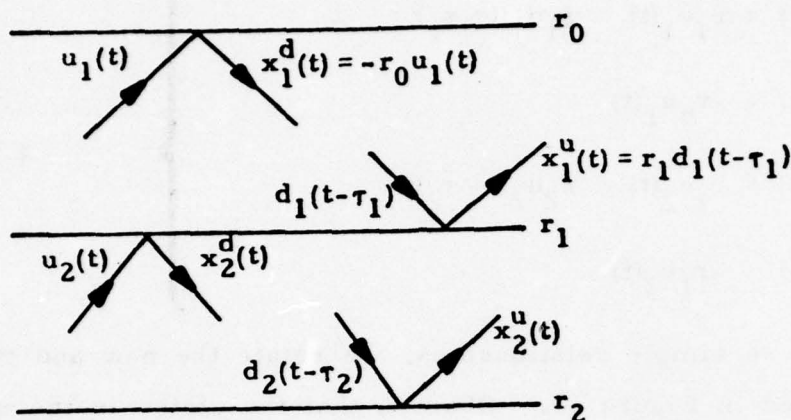


Figure 15. Relationships between signals in the two models.

however, that approach is not very useful in estimation procedures where one would be interested in associating a dynamical model with the primaries. In this section we demonstrate how to modify the complete model in a very simple manner so as to obtain a state space model which generates only the primaries. We refer to this model either as a primary reflection model or as a primaries model.

B. Derivation of Primaries Model

State equations for a primaries model are obtained from Eq. (II-11) by deleting the term $r_1 x_1^d(t)$ in the equation for $x_1^u(t + \tau_1)$, and the term $r_k x_k^d(t)$ in the equations for $x_k^d(t + \tau_k)$, $k = 2, 3, \dots, K$. Our reason for deleting these terms is based on the following observation: a multiple reflection can occur only when an upgoing wave reflects off of the top of a layer. Observe, in Figure 4 that the component x_k^d of μ_k represents just such a reflection, as does the component x_1^d of μ_1 (Figure 6); hence, these components are deleted from the complete model in our primaries model, whose equations are given below.

$$\begin{aligned}
x_{11}^u(t + \tau_1) &= r_1(1 + r_0) m(t) \\
x_{1k}^u(t + \tau_k) &= r_k[(1 + r_{k-1})/r_{k-1}]x_{1,k-1}^u(t) \\
&\quad k = 2, 3, \dots, K \\
x_{1k}^d(t + \tau_k) &= -r_{k-1}x_{1k}^u(t) + r_{k-1}[(1 - r_k)/r_k]x_{1,k+1}^d(t) \\
&\quad k = 1, 2, \dots, K-1 \\
x_{1K}^d(t + \tau_K) &= -r_{K-1}x_{1K}^u(t)
\end{aligned}
\tag{1}$$

and

$$y_1(t) = r_0 m(t) - [(1 - r_0)/r_0]x_{11}^d(t) \tag{2}$$

In these equations x_{1j}^u , x_{1j}^d and y_1 denote primary upgoing and downgoing states and the primary reflection portion of the complete output, respectively.

C. SFG for Primaries Model

An important property of the primaries model is that its SFG has no feedback paths. We illustrate this for a three layer media in Figure 16. This property causes $y_1(t)$ to be comprised of a finite number of terms — $K+1$, for a K layered media, and enables $y_1(t)$ to be computed in a very straightforward manner.

D. Example

In order to validate our primaries model, we consider a three layer media and compute $y_1(t)$ in two different ways: ray tracing and Eqs. (1) and (2). A ray tracing solution is depicted in Figure 17. We assume that the reader is familiar with the details of such a solution; hence, we do not label the amplitudes of the rays on that figure. Signal $y_1(t)$ has four components a, b, c and d which appear in that order in the following expression for $y_1(t)$:

$$\begin{aligned}
y_1(t) &= r_0 m(t) + r_1(1 - r_0^2) m(t - 2\tau_1) \\
&\quad + r_2(1 - r_0^2)(1 - r_1^2) m(t - 2\tau_1 - 2\tau_2)
\end{aligned}$$

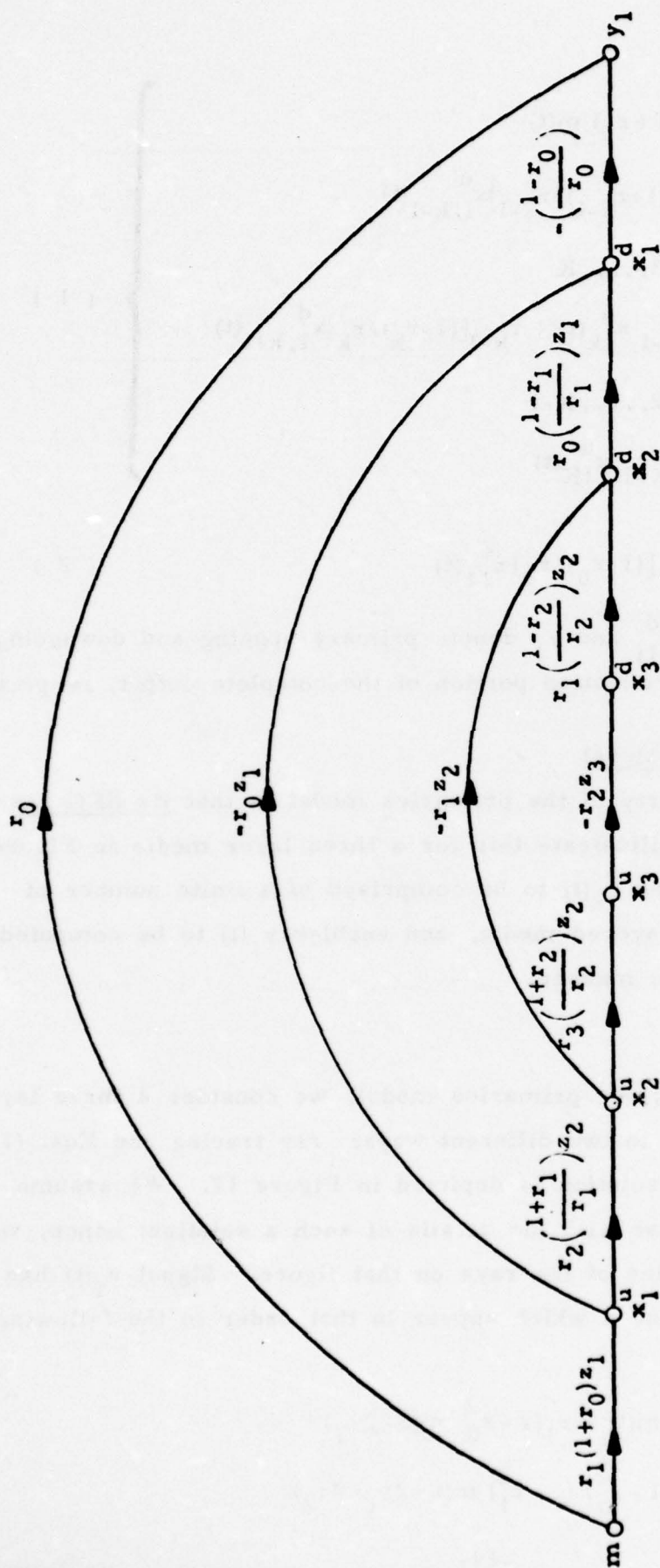


Figure 16. SFG for primaries model for $K=3$.

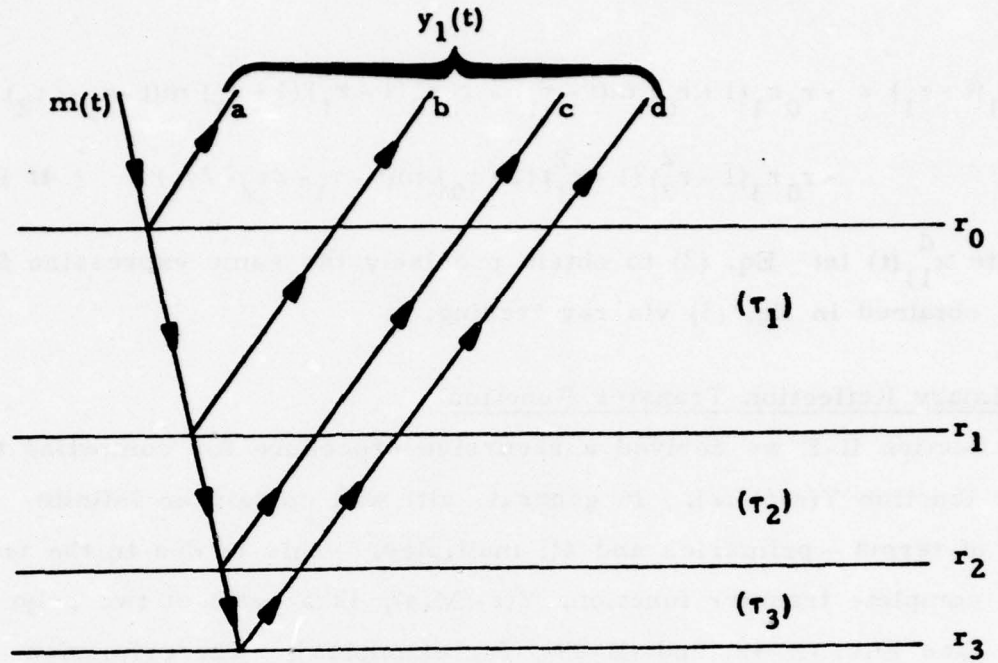


Figure 17. Ray diagram depicting primary reflections as transmitted to the surface.

$$+ r_3(1 - r_0^2)(1 - r_1^2)(1 - r_2^2) m(t - 2\tau_1 - 2\tau_2 - 2\tau_3) \quad (3)$$

The solution for $y_1(t)$ from Eqs. (1) and (2) can be obtained either directly from the Figure 16 SFG or from those equations. Because we shall need some of the states for future calculations in Section V, we shall list the states obtained in order of solution from Eq. (1):

$$x_{11}^u(t + \tau_1) = r_1(1 + r_0) m(t) \quad (4a)$$

$$x_{12}^u(t + \tau_2) = r_2(1 + r_0)(1 + r_1) m(t - \tau_1) \quad (4b)$$

$$x_{13}^u(t + \tau_3) = r_3(1 + r_2)(1 + r_1)(1 + r_0) m(t - \tau_1 - \tau_2) \quad (4c)$$

$$x_{13}^d(t + \tau_3) = -r_2 r_3(1 + r_2)(1 + r_1)(1 + r_0) m(t - \tau_1 - \tau_2 - \tau_3) \quad (4d)$$

$$x_{12}^d(t + \tau_2) = -r_1 r_2(1 + r_1)(1 + r_0) m(t - \tau_1 - \tau_2) \\ - r_1 r_3(1 - r_2^2)(1 + r_1)(1 + r_0) m(t - \tau_1 - \tau_2 - 2\tau_3) \quad (4e)$$

$$\begin{aligned}
x_{11}^d(t+\tau_1) = & -r_0 r_1 (1+r_0) m(t-\tau_1) - r_0 r_2 (1-r_1^2)(1+r_0) m(t-\tau_1-2\tau_2) \\
& - r_0 r_3 (1-r_2^2)(1-r_1^2)(1+r_0) m(t-\tau_1-2\tau_2-2\tau_3) \quad (4f)
\end{aligned}$$

Substitute $x_{11}^d(t)$ into Eq. (2) to obtain precisely the same expression for $y_1(t)$ as obtained in Eq. (3) via ray tracing.

E. Primary Reflection Transfer Function

In Section II-E we derived a recursive procedure for computing the transfer function $Y(s)/M(s)$. In general, $y(t)$ will contain an infinite number of terms — primaries and all multiples. This is due to the fact that the complete transfer function, $Y(s)/M(s)$, is a ratio of two polynomials [see Eqs. (II-48) and (II-49), for examples]. The primaries transfer function, $Y_1(s)/M(s)$, on the other hand has no poles. This is evident, for example, from the example in Figure 16.

We do not wish to redevelop the recursive procedure for computing $Y_1(s)/M(s)$ in the detail of Section II-E. The results will look exactly like those given in Eqs. (II-43) and (II-36), except that all matrices and vectors should have an additional subscript 1 (e.g., W_{1K}, Y_1) to associate them with the primaries model. An important difference occurs in the definition of $F_{1k}(s)$:

$$F_{1k}(s) = \begin{pmatrix} e^{s\tau_k} & 0 \\ r_{k-1} & e^{s\tau_k} \end{pmatrix} \quad (5)$$

Matrices $F_{1k}(s)$ are all lower triangular; and, one can therefore demonstrate, by direct calculations, that matrices $W_{1,K-1}, W_{1,K-2}, \dots, W_{12}, W_{11}$ are also lower triangular; hence, W_{11}^{-1} is lower triangular, say

$$W_{11}^{-1}(s) = \begin{pmatrix} \alpha(s) & 0 \\ \beta(s) & \gamma(s) \end{pmatrix} \quad (6)$$

From Eqs. (II-43) and (6), we see that

$$\frac{Y_1(s)}{M(s)} = r_0 - (0 \ 1) W_{11}^{-1}(s) \begin{pmatrix} r_1(1-r_0^2)/r_0 \\ 0 \end{pmatrix}$$

$$\frac{Y_1(s)}{M(s)} = r_0 - \frac{r_1(1-r_0^2)}{r_0} \beta(s) \quad (7)$$

which proves our assertion that the primaries transfer function has no poles. We shall have more to say about this property of the primaries model in Section VI.

V. A SECONDARY REFLECTION MODEL

A. Introduction

In Section IV we developed a primaries model which permits us to remove the primary reflections from the complete output. We are interested in learning whether it is possible to develop a secondary reflection model (or, secondaries model) which would generate just the secondary reflections. Secondary reflections, which are the first multiple reflections, are components of the complete output which are due to exactly three reflections within a K layer media system. In some applications the secondaries can be significant contributors to the complete output; whereas, in other applications they may be of such small amplitude (due to the product of three reflection coefficients in each secondary) that they can be neglected. In both situations it is of interest to establish whether they should or should not be accounted for. The secondaries model which we develop in this section will permit one to focus his attention just at the secondary reflections; hence, he will be able to establish the significance of the secondaries without the encumbrance of primaries and other multiples, which are present in the complete output.

Let $x_{2j}^u(t)$ and $x_{2j}^d(t)$ denote secondary upgoing and downgoing states in the j^{th} layer ($j=1, 2, \dots, K$), and, let $y_2(t)$ denote the secondary reflection portion of the complete output. Our objective in this section is to develop state equations for $x_{2j}^u(t)$ and $x_{2j}^d(t)$ and the equation for $y_2(t)$.

B. A First Residual Model

To begin, we form a residual model between our complete model and primaries model. Subtract the respective equations in Eqs. (II-11) and (IV-1) to obtain:

$$x_1^u(t + \tau_1) - x_{11}^u(t + \tau_1) = r_1 x_1^d(t) \quad (1a)$$

$$x_k^u(t + \tau_k) - x_{1k}^u(t + \tau_k) = r_k x_k^d(t) + r_k [(1+r_{k-1})/r_{k-1}] \\ \cdot [x_{k-1}^u(t) - x_{1,k-1}^u(t)] \quad (1b)$$

$$k = 2, 3, \dots, K$$

$$x_k^d(t + \tau_k) - x_{1k}^d(t + \tau_k) = -r_{k-1} [x_k^u(t) - x_{1k}^u(t)] \\ + r_{k-1} [(1-r_k)/r_k] [x_{k+1}^d(t) - x_{1,k+1}^d(t)] \quad (1c)$$

$$k = 1, 2, \dots, K-1$$

$$x_K^d(t + \tau_K) - x_{1K}^d(t + \tau_K) = -r_{K-1} [x_K^u(t) - x_{1K}^u(t)] \quad (1d)$$

Additionally, subtract $y_1(t)$ [Eq. (IV-2)] from $y(t)$ [Eq. (II-12)], to obtain

$$y(t) - y_1(t) = -[(1-r_0)/r_0] [x_1^d(t) - x_{11}^d(t)] \quad (2)$$

Next, we define first residual states $\xi_{1j}^u(t)$ and $\xi_{1j}^d(t)$ and first residual output, $s_1(t)$, as

$$\xi_{1j}^u(t) = x_j^u(t) - x_{1j}^u(t) \quad (3a)$$

$$\xi_{1j}^d(t) = x_j^d(t) - x_{1j}^d(t) \quad (3b)$$

and

$$s_1(t) = y(t) - y_1(t) \quad (4)$$

Adding and subtracting the term $r_1 x_{11}^d(t)$ from Eq. (1a) and $r_k x_{1k}^d(t)$ from Eq. (1b), and substituting Eqs. (3a) and (3b) into Eqs. (1a)-(1d), we obtain the following first residual state equations:

$$\begin{aligned}
\xi_{11}^u(t+\tau_1) &= r_1 \xi_{11}^d(t) + r_1 x_{11}^d(t) \\
\xi_{1k}^u(t+\tau_k) &= r_k \xi_{1k}^d(t) + r_k [(1+r_{k-1})/r_{k-1}] \xi_{1,k-1}^u(t) + r_k x_{1k}^d(t) \\
k &= 2, 3, \dots, K \\
\xi_{1k}^d(t+\tau_k) &= -r_{k-1} \xi_{1k}^u(t) + r_{k-1} [(1-r_k)/r_k] \xi_{1,k+1}^d(t) \\
\xi_{1K}^d(t+\tau_K) &= -r_{K-1} \xi_{1K}^u(t)
\end{aligned}
\tag{5}$$

Additionally, the first residual output, $s_1(t)$, is

$$s_1(t) = -[(1-r_0)/r_0] \xi_{11}^d(t) \tag{6}$$

At this point some important observations are in order. By its very definition, $s_1(t)$ comprises all the multiple reflections for a K layer media system. The first residual model (5) and (6) can therefore be used to generate the complete set of multiple reflections — secondaries, tertiaries, etc. Observe also that the first residual state equations are driven by the downgoing states from the primaries model; hence, the primaries model is coupled into the first residuals model, but in an open-loop manner.

C. Secondaries Model

Compare state equations (5) with state equations (II-11), to observe that they are structurally quite similar. Recall that in the development of the primaries model we deleted the terms $r_1 x_{11}^d(t)$ and $r_k x_{1k}^d(t)$ from the x_k^u state equations. We conjecture, therefore, that the state equations for the secondaries model are obtained from Eq. (5) by deleting the term $r_1 \xi_{11}^d(t)$ in the equation for $\xi_{11}^u(t+\tau_1)$, and the term $r_k \xi_{1k}^d(t)$ in the equations for $\xi_{1k}^u(t+\tau_k)$, $k=2, 3, \dots, K$. The resulting secondaries model is:

$$\begin{aligned}
x_{21}^u(t+\tau_1) &= r_1 x_{11}^d(t) \\
x_{2k}^u(t+\tau_k) &= r_k [(1+r_{k-1})/r_{k-1}] x_{2,k-1}^u(t) + r_k x_{1k}^d(t) \\
k &= 2, 3, \dots, K
\end{aligned}$$

$$\begin{aligned}
 x_{2k}^d(t + \tau_k) &= -r_{k-1} x_{2k}^u(t) + r_{k-1} [(1-r_k)/r_k] x_{2,k+1}^d(t) \\
 k &= 1, 2, \dots, K-1 \\
 x_{2K}^d(t + \tau_K) &= -r_{K-1} x_{2K}^u(t)
 \end{aligned}
 \tag{7}$$

and

$$y_2(t) = -[(1-r_0)/r_0] x_{21}^d(t) \tag{8}$$

A proof of this result will appear in a forthcoming paper. We validate it by means of an example in Paragraph E. We summarize the essence of the secondaries model in Figure 18. Observe, again, the dependence of the secondaries model on all of the downgoing states from the primaries model. Additional discussions on the structure of the block diagram in Figure 18 are given in Section VI.

D. SFG for Secondaries Model

An important property of the secondaries model is that its SFG has no feedback paths. We illustrate this for a three layer media in Figure 19. This property causes $y_2(t)$ to be comprised of a finite number of terms and enables $y_2(t)$ to be computed in a very straightforward manner.

E. Example

To validate our secondaries model, we consider a three layer media and compute $y_2(t)$ in two different ways: ray tracing and Eqs. (7) and (8).

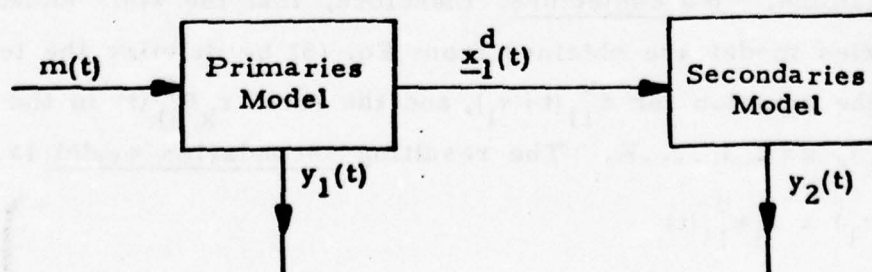


Figure 18. Generation of primaries, $y_1(t)$, and secondaries, $y_2(t)$.

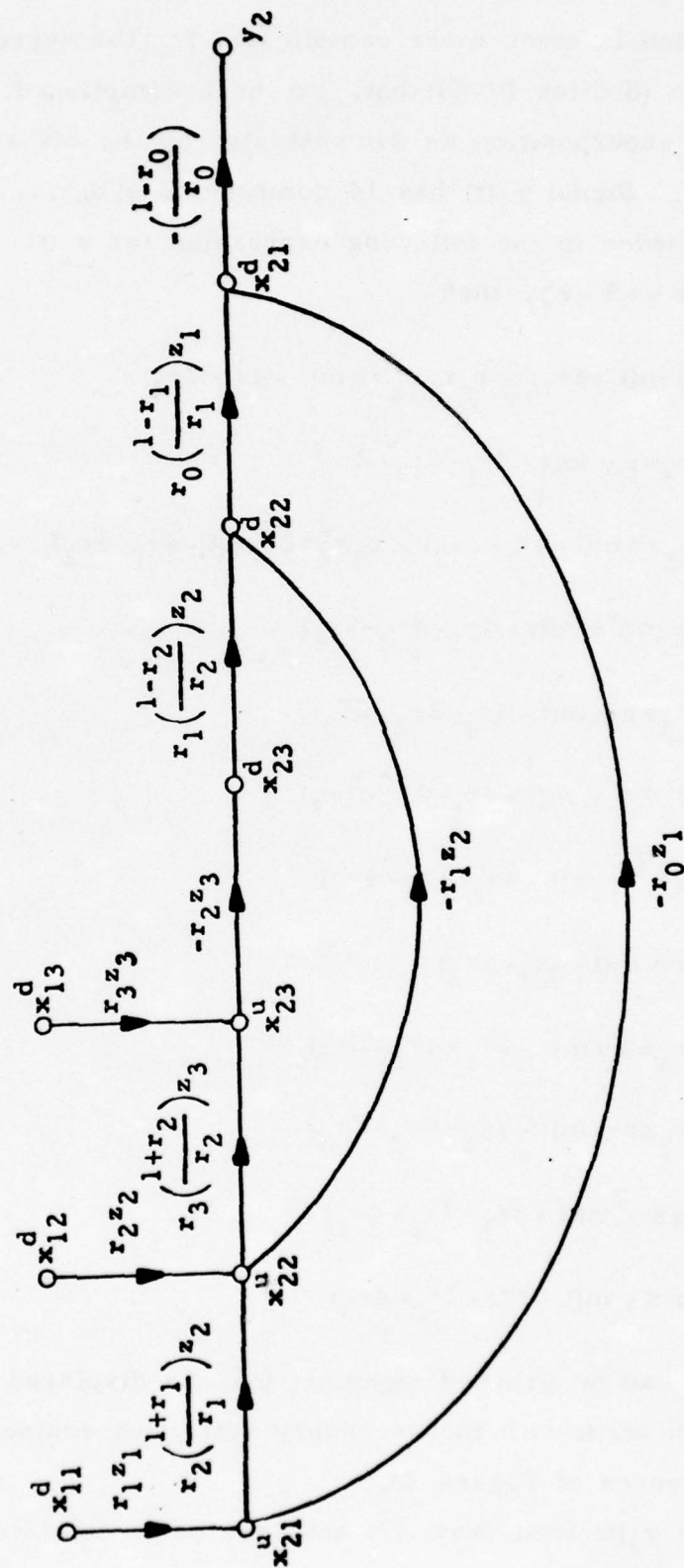


Figure 19. SFG for secondaries model for $K=3$.

The ray tracing solution is much more complicated for the secondaries than for the primaries (Section IV-C); but, can be accomplished for the three layer media by superposition as demonstrated by the six situations depicted in Figure 20. Signal $y_2(t)$ has 14 components a, b, c, ..., and n which appear in that order in the following expression for $y_2(t)$: let $\alpha = 1 - r_0^2$, $\beta = 1 - r_1^2$ and $\gamma = 1 - r_2^2$, then

$$\begin{aligned}
 y_2(t) = & -r_0 r_1^2 \alpha m(t - 4\tau_1) - r_0 r_1 r_2 \alpha \beta m(t - 4\tau_1 - 2\tau_2) \\
 & - r_0 r_1 r_3 \alpha \beta \gamma m(t - 4\tau_1 - 2\tau_2 - 2\tau_3) \\
 & - r_0 r_1 r_2 \alpha \beta m(t - 4\tau_1 - 2\tau_2) - r_0 r_2^2 \alpha \beta^2 m(t - 4\tau_1 - 4\tau_2) \\
 & - r_0 r_3 r_2 \alpha \beta^2 \gamma m(t - 4\tau_1 - 4\tau_2 - 2\tau_3) \\
 & - r_0 r_1 r_3 \alpha \beta \gamma m(t - 4\tau_1 - 2\tau_2 - 2\tau_3) \\
 & - r_0 r_2 r_3 \alpha \beta^2 \gamma m(t - 4\tau_1 - 4\tau_2 - 2\tau_3) \\
 & - r_0 r_3^2 \alpha \beta^2 \gamma^2 m(t - 4\tau_1 - 4\tau_2 - 4\tau_3) \\
 & - r_1 r_2^2 \alpha \beta m(t - 2\tau_1 - 4\tau_2) \\
 & - r_1 r_2 r_3 \alpha \beta \gamma m(t - 2\tau_1 - 4\tau_2 - 2\tau_3) \\
 & - r_1 r_2 r_3 \alpha \beta \gamma m(t - 2\tau_1 - 4\tau_2 - 2\tau_3) \\
 & - r_1 r_3^2 \alpha \beta \gamma^2 m(t - 2\tau_1 - 4\tau_2 - 4\tau_3) \\
 & - r_2 r_3^2 \alpha \beta \gamma m(t - 2\tau_1 - 2\tau_2 - 4\tau_3)
 \end{aligned} \tag{9}$$

Many of these terms can be grouped together; but, as displayed we are able to correlate each term with the secondary reflection components depicted in the six figures of Figure 20.

The solution for $y_2(t)$ from Eqs. (7) and (8) can be obtained directly

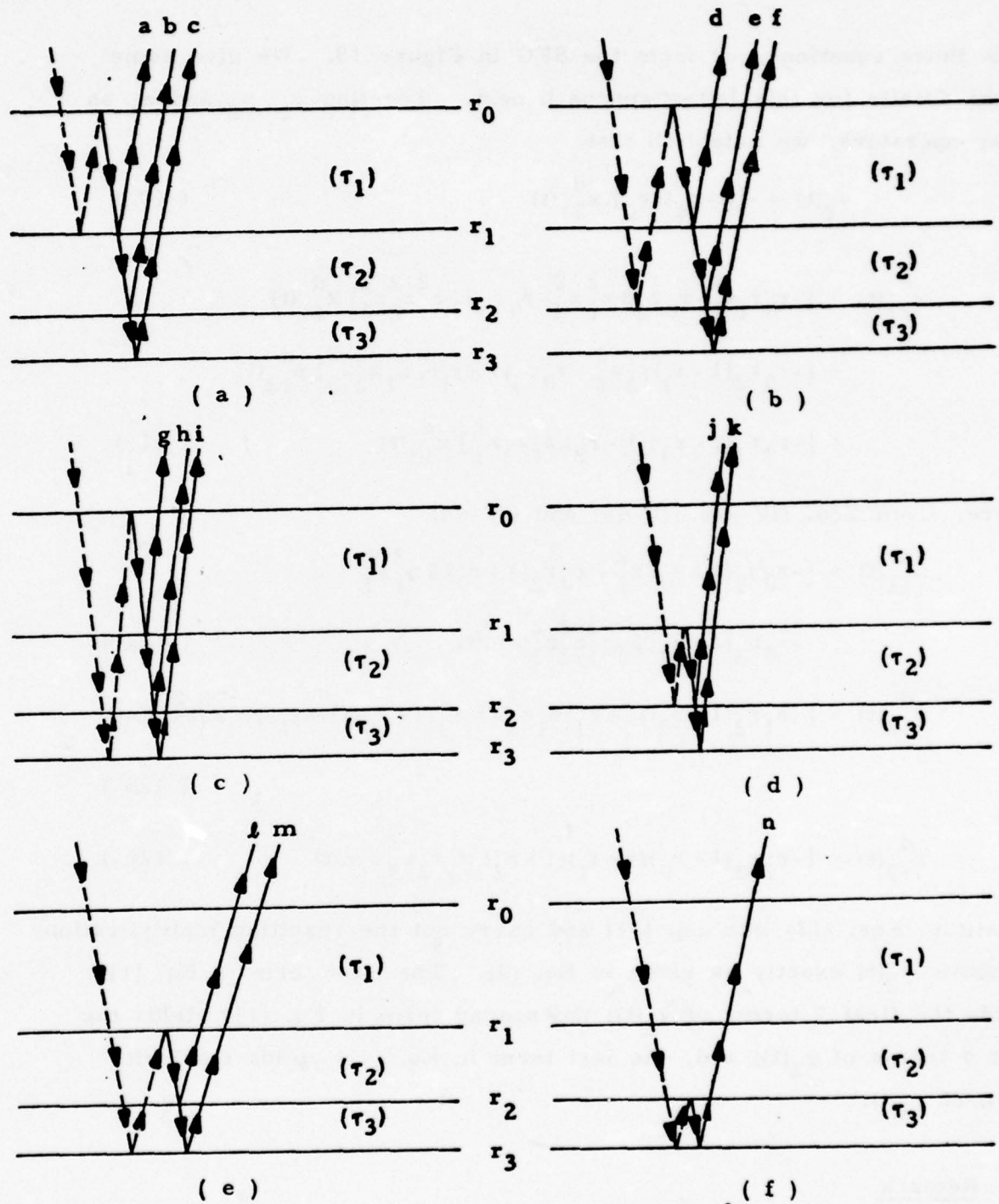


Figure 20. Ray diagrams depicting secondary reflections as transmitted to the surface. Dashed rays indicate portion of a primary reflection which generates subsequent secondary reflections.

from those equations, or from the SFG in Figure 19. We give some of the details for this latter approach next. Treating z_1 , z_2 and z_3 as delay operators, we establish that

$$y_2(t) = -[(1-r_0)/r_0] x_{21}^d(t) \quad (10)$$

and

$$\begin{aligned} x_{21}^d(t) = & [-r_0 r_1 z_1^2 - r_0 r_2 \beta z_1^2 z_2^2 - r_0 r_3 \beta \gamma z_1^2 z_2^2 z_3^2] x_{11}^d(t) \\ & + [-r_0 r_2 (1-r_1) z_1 z_2^2 - r_0 r_3 (1-r_1) \gamma z_1 z_2^2 z_3^2] x_{12}^d(t) \\ & + [-r_0 r_3 (1-r_1)(1-r_2) z_1 z_2 z_3^2] x_{13}^d(t) \end{aligned} \quad (11)$$

where, from Eqs. (IV-4f), (IV-4e) and (IV-4d),

$$\begin{aligned} x_{11}^d(t) = & [-r_0 r_1 (1+r_0) z_1^2 - r_0 r_2 (1+r_0) \beta z_1^2 z_2^2 \\ & - r_0 r_3 (1+r_0) \beta \gamma z_1^2 z_2^2 z_3^2] m(t) \end{aligned} \quad (12a)$$

$$x_{12}^d(t) = [-r_1 r_2 (1+r_0)(1+r_1) z_1 z_2^2 - r_1 r_3 (1+r_0)(1+r_1) \gamma z_1 z_2^2 z_3^2] m(t) \quad (12b)$$

and

$$x_{13}^d(t) = [-r_2 r_3 (1+r_0)(1+r_1)(1+r_2) z_1 z_2 z_3^2] m(t) \quad (12c)$$

Substitute Eqs. (12) into Eq. (11) and carry out the resulting multiplications to obtain $y_2(t)$ exactly as given in Eq. (9). The first term in Eq. (11) yields the first 9 terms of $y_2(t)$; the second term in Eq. (11) yields the next 4 terms of $y_2(t)$; and, the last term in Eq. (11) yields the 14th term of $y_2(t)$.

F. Remark

It is only in simple examples which involve a small number of layers that one can obtain $y_2(t)$ via ray tracing techniques. There are just too many possibilities for enumeration in larger systems. However, our secondaries state space model is applicable regardless of how many layers are present. We are presently studying computer implementations of these equations.

VI. A COMPLETE DECOMPOSITION OF A SEISMOGRAM

Our results in the two preceding sections suggest that a complete decomposition of a seismogram [i.e., $y(t)$], into a superposition of outputs from primaries, secondaries, tertiaries, etc. models is possible. We will not go into all of the details of such a decomposition here; but, instead shall sketch its development.

To begin, let us sketch the development of a tertiaries model. Tertiary reflections are the second collection of multiple reflections, and are those components of the complete output which are due to exactly five reflections within a K layer media system. To obtain a tertiaries model, we proceed exactly as in our development of the secondaries model. First, we form a second residual model between the first residual model in Eqs. (V-5) and (V-6) and the secondaries model in Eqs. (V-7) and (V-8). The second residual state variables are $\xi_{2j}^u(t)$ and $\xi_{2j}^d(t)$, where

$$\xi_{2j}^u(t) = \xi_{1j}^u(t) - x_{2j}^u(t) \quad (1a)$$

and

$$\xi_{2j}^d(t) = \xi_{1j}^d(t) - x_{2j}^d(t) \quad (1b)$$

for $j = 1, 2, \dots, K$, and, the second residual output is $s_2(t)$, where

$$s_2(t) = s_1(t) - y_2(t) \quad (2)$$

The second residual state equations will look exactly like the first residual state equations in (V-5) with ξ_{1j}^u and ξ_{1j}^d replaced by ξ_{2j}^u and ξ_{2j}^d , respectively, and x_{1l}^d and x_{1k}^d replaced by x_{2l}^d and x_{2k}^d , respectively. The second residual state equations are driven by the downgoing components of the secondaries state vector.

The tertiaries model is obtained from the second residual model by deleting the terms $r_1 x_{2l}^d(t)$ and $r_k x_{2k}^d(t)$ in the $\xi_{2l}^u(t + \tau_1)$ and $\xi_{2k}^u(t + \tau_k)$ state equations. The resulting tertiaries model is:

$$\left. \begin{aligned}
 x_{31}^u(t + \tau_1) &= r_1 x_{21}^d(t) \\
 x_{3k}^u(t + \tau_k) &= r_k [(1+r_{k-1})/r_{k-1}] x_{3,k-1}^u(t) + r_k x_{2k}^d(t) \\
 k &= 2, 3, \dots, K \\
 x_{3k}^d(t + \tau_k) &= -r_{k-1} x_{3k}^u(t) + r_{k-1} [(1-r_k)/r_k] x_{3,k+1}^u(t) \\
 x_{3K}^d(t + \tau_K) &= -r_{K-1} x_{3K}^u(t)
 \end{aligned} \right\} \quad (3)$$

and

$$y_3(t) = -[(1-r_0)/r_0] x_{31}^d(t) \quad (4)$$

Observe that the tertiaries model is driven by the downgoing components of the secondaries state vector. Figure 18 can now be expanded to include the tertiaries model driven by $x_2^d(t)$ with output $y_3(t)$.

In this manner we can form successive residual models and subsequent n-aries models, each model being of dimension $2K$. The following complete (canonical) decomposition of $y(t)$ has been demonstrated by the second author of this paper:

The complete output, $y(t)$, from a K layer media system, which is comprised of the superposition of primaries, secondaries, tertiaries, etc., can be obtained from a single model of order $2K$ — the complete model — or from an infinite number of models, each of order $2K$, interconnected as shown in Figure 21.

There are many interesting implications and potentially useful applications for this decomposition. In seismic applications, where reflection coefficients are often quite small [$|r_i| < 0.3$], the decomposition could be used as a computational tool for approximating the complete seismogram $y(t)$ by a small collection of constituent reflections, such as $y_1(t) + y_2(t)$. Each one of the constituent models in Figure 21 is a non-feedback model, whereas the complete model is a very complicated feedback system; hence, computation via a truncated decomposition of $y(t)$ may be quite expeditious. Additionally, by means of the canonical

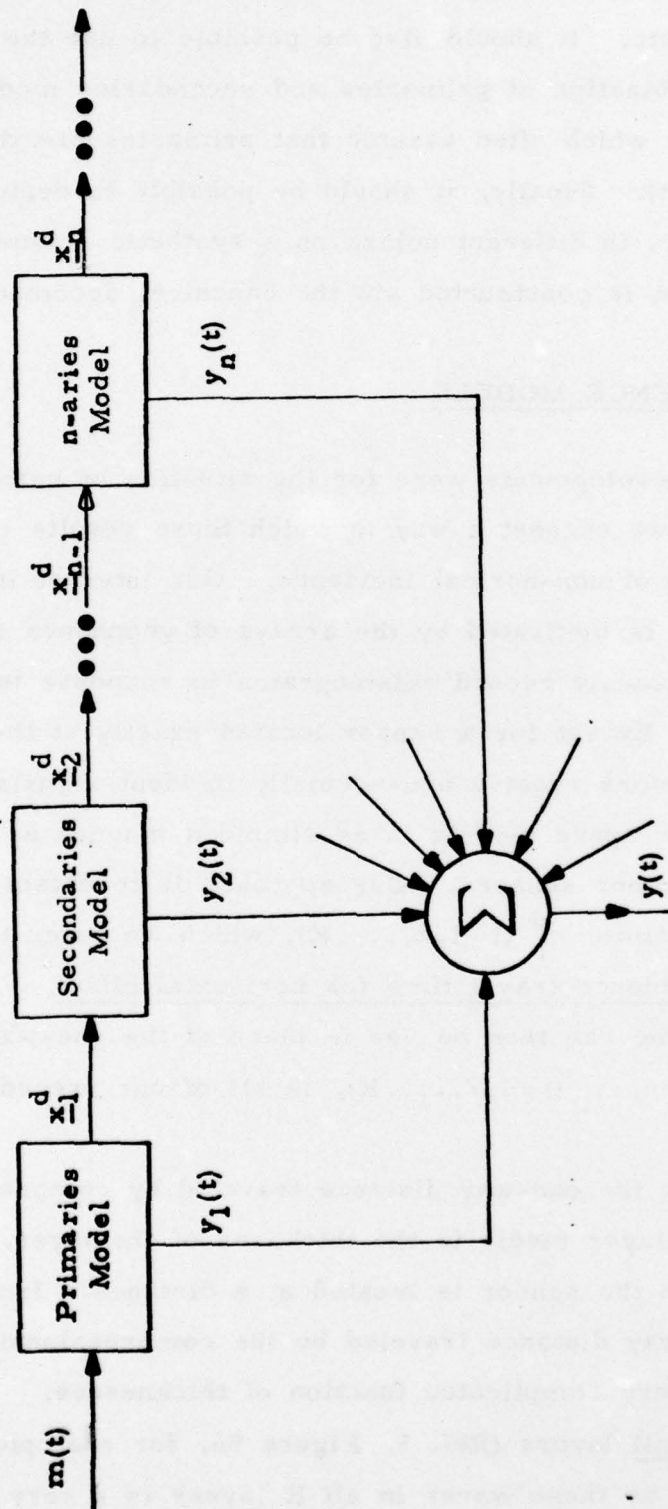


Figure 21. Canonical decomposition of a seismogram signal, $y(t)$.

decomposition it should be easy to compare the relative importance of secondaries to primaries, etc. It should also be possible to use the primaries model (or a combination of primaries and secondaries models) in deconvolution techniques, which often assume that primaries are the only significant terms in $y(t)$. Finally, it should be possible to depict primaries, secondaries, etc. in different colors on a synthetic seismogram, when the seismogram is constructed via the canonical decomposition.

VII. NON-NORMAL INCIDENCE MODELS

All of the preceding developments were for the situation of normal incidence. In this section we suggest a way in which these results can be extended to the situation of non-normal incidence. Our interest in obtaining such an extension is motivated by the arrays of geophones or hydrophones which simultaneously record seismograms in response to a seismic source signature. Except for a sensor located exactly at the source point, the other sensors receive non-normally incident signals. We wish to extend our state space models in as simple a manner as possible to describe these other sensors. Our approach is to obtain an expression for a travel time, τ_i^d ($i=1,2,\dots,K$), which we refer to as one-way non-normal incidence travel time for horizontal offset distance d . This travel time can then be used in place of the one-way normal incidence travel time, τ_i ($i=1,2,\dots,K$), in all of our preceding models.

For normal incidence, the one-way distance traveled by compressional waves in any layer of a K layer media is the thickness of the layer. For non-normal incidence, when the sensor is located at a distance d from the source point, the one-way distance traveled by the compressional waves in each layer is a very complicated function of thicknesses, densities and velocities of all layers (Ref. 5, Figure 96, for example). The total distance traveled by these waves in all K layers is a very

complicated function of all layer thicknesses and all angles made by each section of the wave path (Figure 22).

Our approach is to approximate the one-way primary reflection paths by straight lines drawn from the source point to the point $d/2$ at the bottom of each layer (Figure 23). From the geometry in Figure 23,

$$\tau_i^d = l_i / V_i \quad (1)$$

and

$$\tau_i = h_i / V_i \quad (2)$$

so that

$$\tau_i^d = \tau_i (l_i / h_i) \quad ; \quad (3)$$

but,

$$l_i / h_i = 1 / \cos \theta_i = [1 + \tan^2 \theta_i]^{1/2} = [1 + (d/2)^2 / \sum_{j=1}^i h_j]^{1/2} \quad (4)$$

which can also be written, from Eq. (2), as

$$l_i / h_i = [1 + (d/2)^2 / \sum_{j=1}^i \tau_j V_j]^{1/2} \quad (5)$$

Our final expression for τ_i^d is obtained from Eqs. (3) and (5), as

$$\tau_i^d = \tau_i [1 + d^2 / 4 \sum_{j=1}^i \tau_j V_j]^{1/2} \quad (6)$$

Observe that τ_i^d is an increasing function of d , that for $d=0$, $\tau_i^d = \tau_i$, and, that τ_i^d becomes closer and closer to τ_i for deep layers.

We propose to make use of τ_i^d in the following manner. Replace τ_i in all of our earlier models by τ_i^d in order to compute the signal observed by a sensor which is offset d units from the source. The implications of this replacement for multiple reflections remains to be studied.

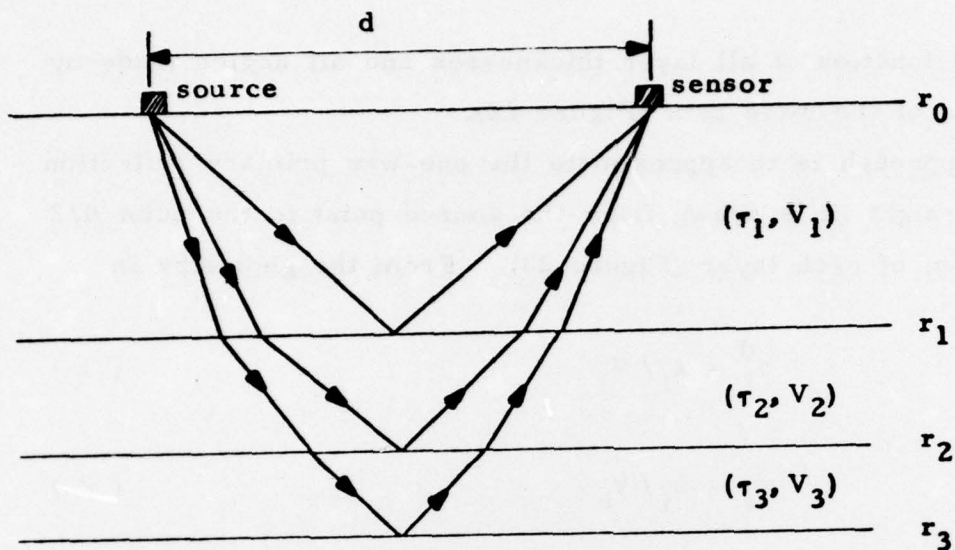


Figure 22. Some non-normal incident primary reflection paths taken by waves in reaching a sensor offset d units from the source.

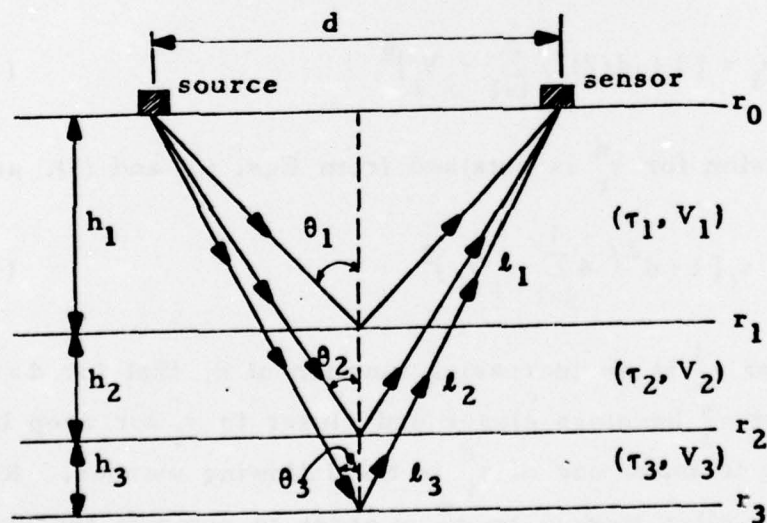


Figure 23. Some straight line approximations to non-normal incident primary reflection paths.

VIII. CONCLUSIONS

In this paper we have developed state space models for a K layer media system, and, have shown how to obtain transfer functions from these models. Our models have been derived by examining wave effects within a single layer, are time-domain models, and do not assume equal travel times in each layer.

Our complete model is of order $2K$; it generates primaries and all multiples, and has an enormous amount of internal feedback structure. Our primaries and secondaries models are each of order $2K$; the former generates just the primary reflection components of a seismogram, whereas the latter generates just the secondary reflection components of a seismogram. Both the primaries and secondaries models have no internal feedback structure.

Our development of the primaries and secondaries models has led us to a complete (canonical) decomposition of a seismogram (see Figure 21). The utility of this decomposition remains to be explored.

Finally, we have shown how our normal incidence models can be extended to the non-normal incidence situation.

There are many interesting directions for future work, including: (1) computational aspects of the various models; (2) implications and applications of the complete (canonical) decomposition of a seismogram; (3) understanding the system theoretic aspects of causal functional equations; (4) developing optimal state estimators for our models; (5) studying parameter identification problems for these models, etc.

In conclusion, we also point out that, whereas we have presented all of our results in the context of a geophysical model, most of them (with the possible exclusion of the material in Section VII) are applicable to any application which is adequately modeled by a lossless wave equation in which ray theory solutions are utilized. We shall explore the more general nature of our results in a later paper.

ACKNOWLEDGEMENT

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Simulation Results

Simulation results, which were obtained after the paper was prepared, are given in the following figures.

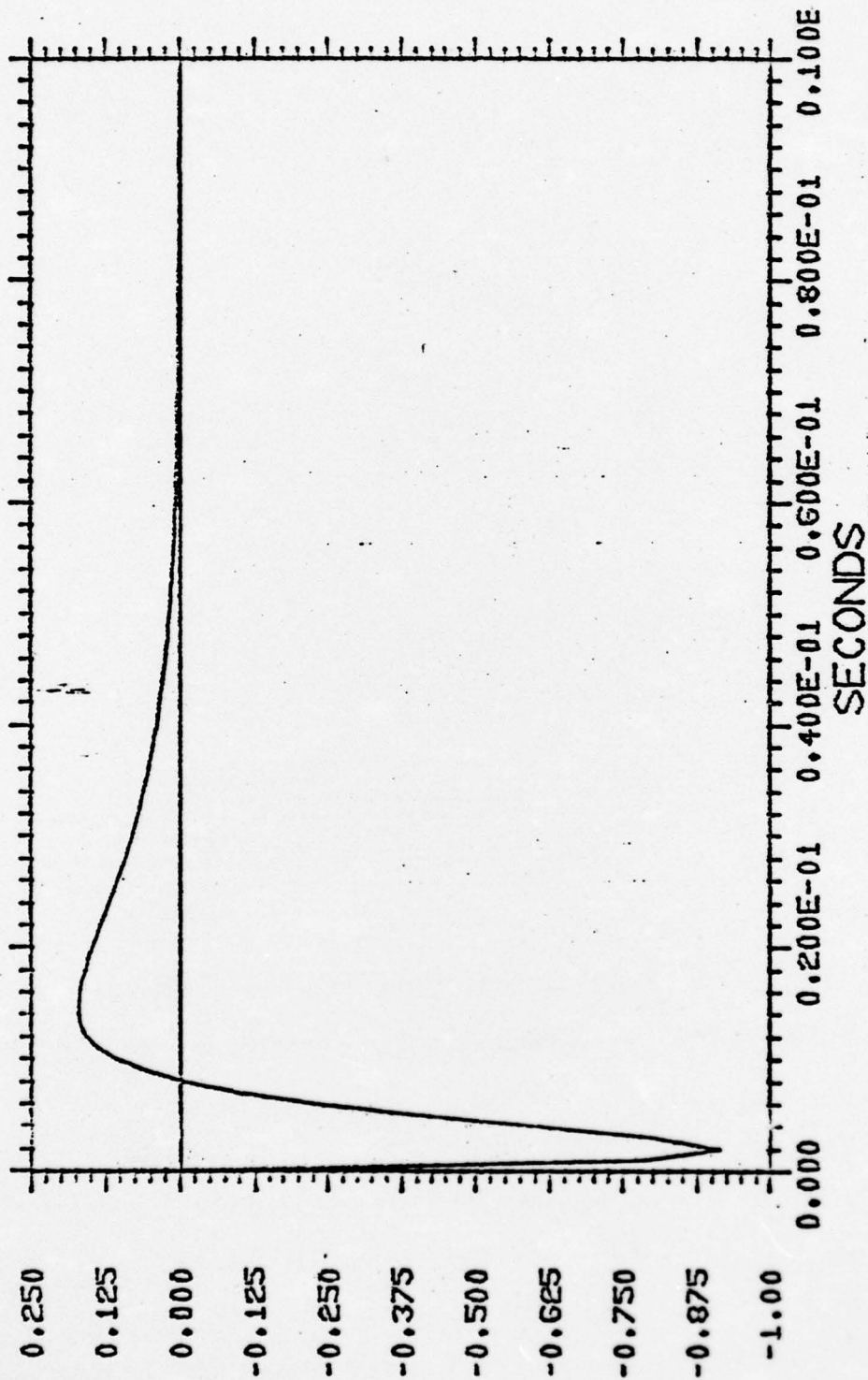


Figure 24. A fourth-order seismic wavelet (Ref. "Seismic Energy Sources 1968 Handbook", F.S. Kramer et al (eds.), Presented at the 38th Annual Meeting of the SEG, Denver, Colorado, October 1968).

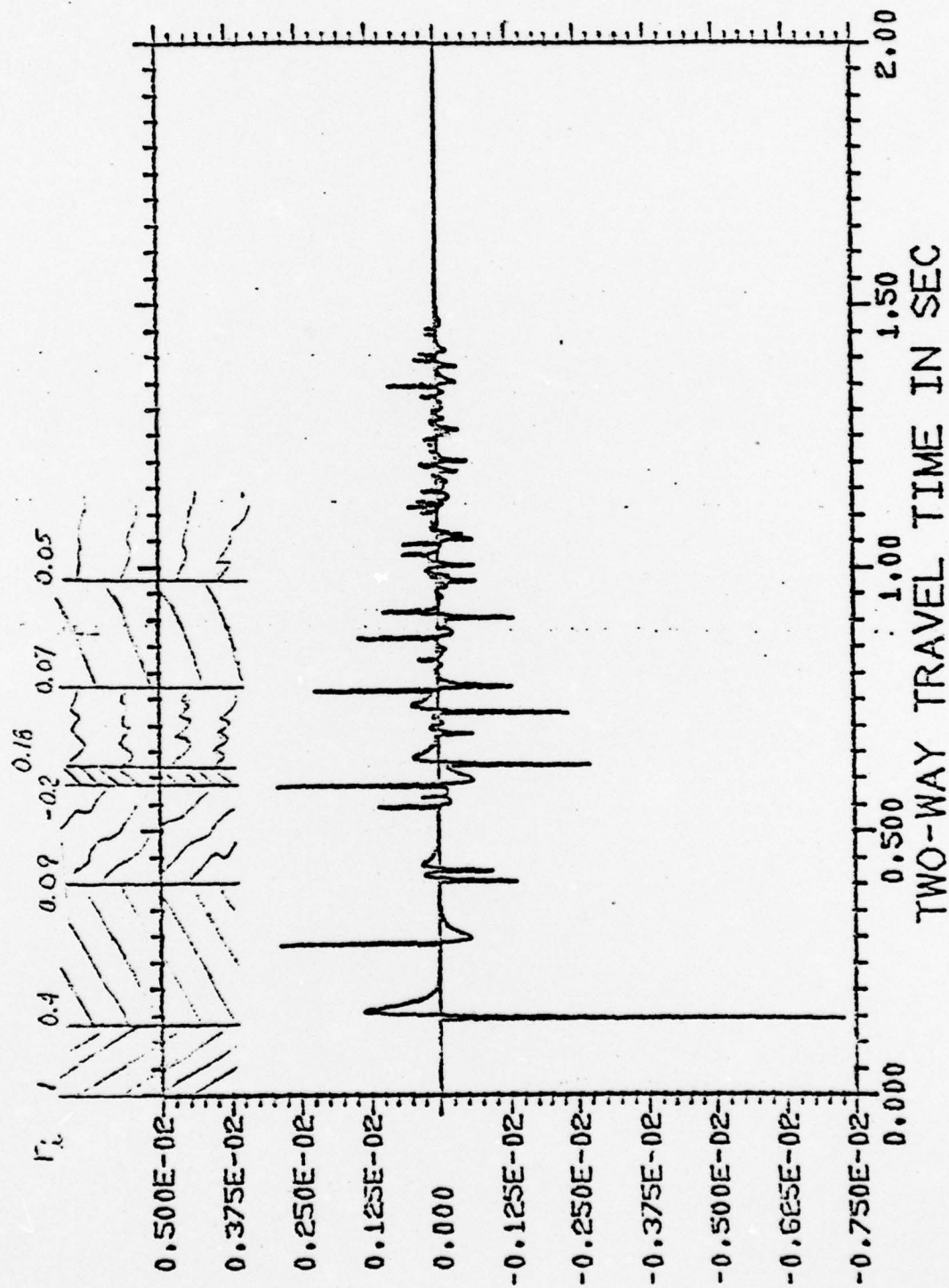


Figure 25. Complete response of a six-layer media. The layered media is depicted at the top of the figure. Source and sensor are on the surface.

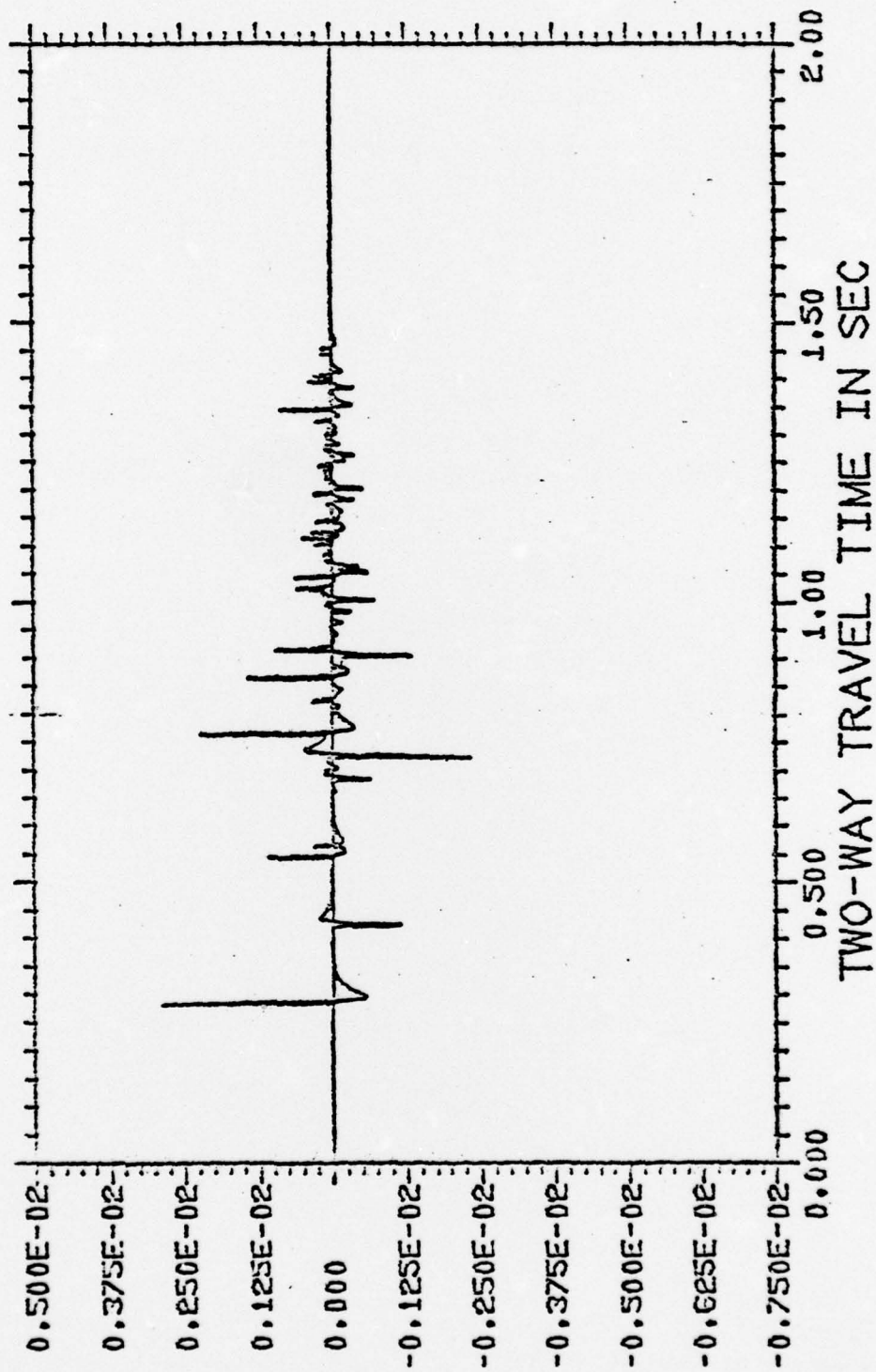


Figure 26. First residual (i.e., complete response minus the primaries)

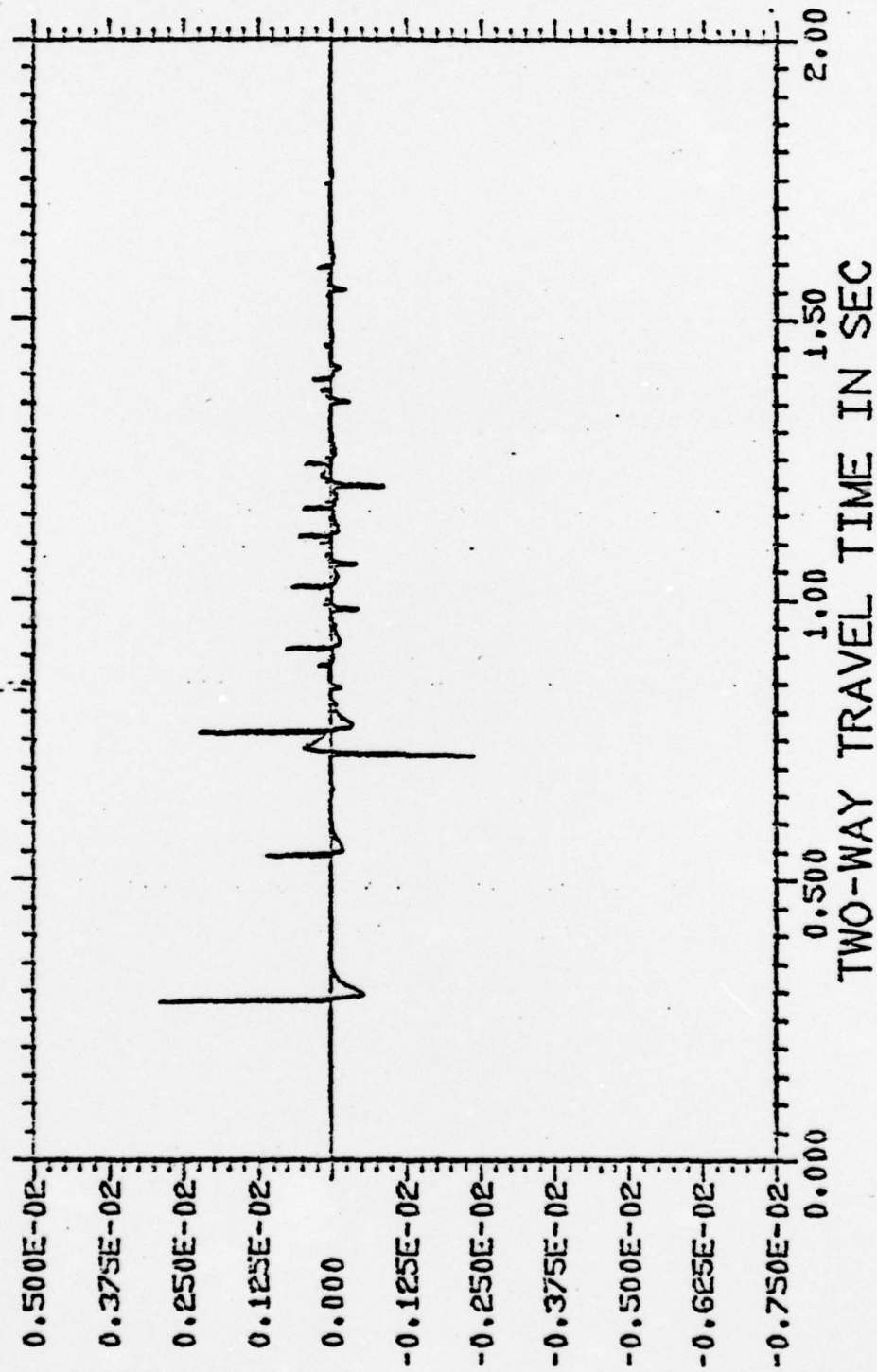


Figure 27. Secondaries

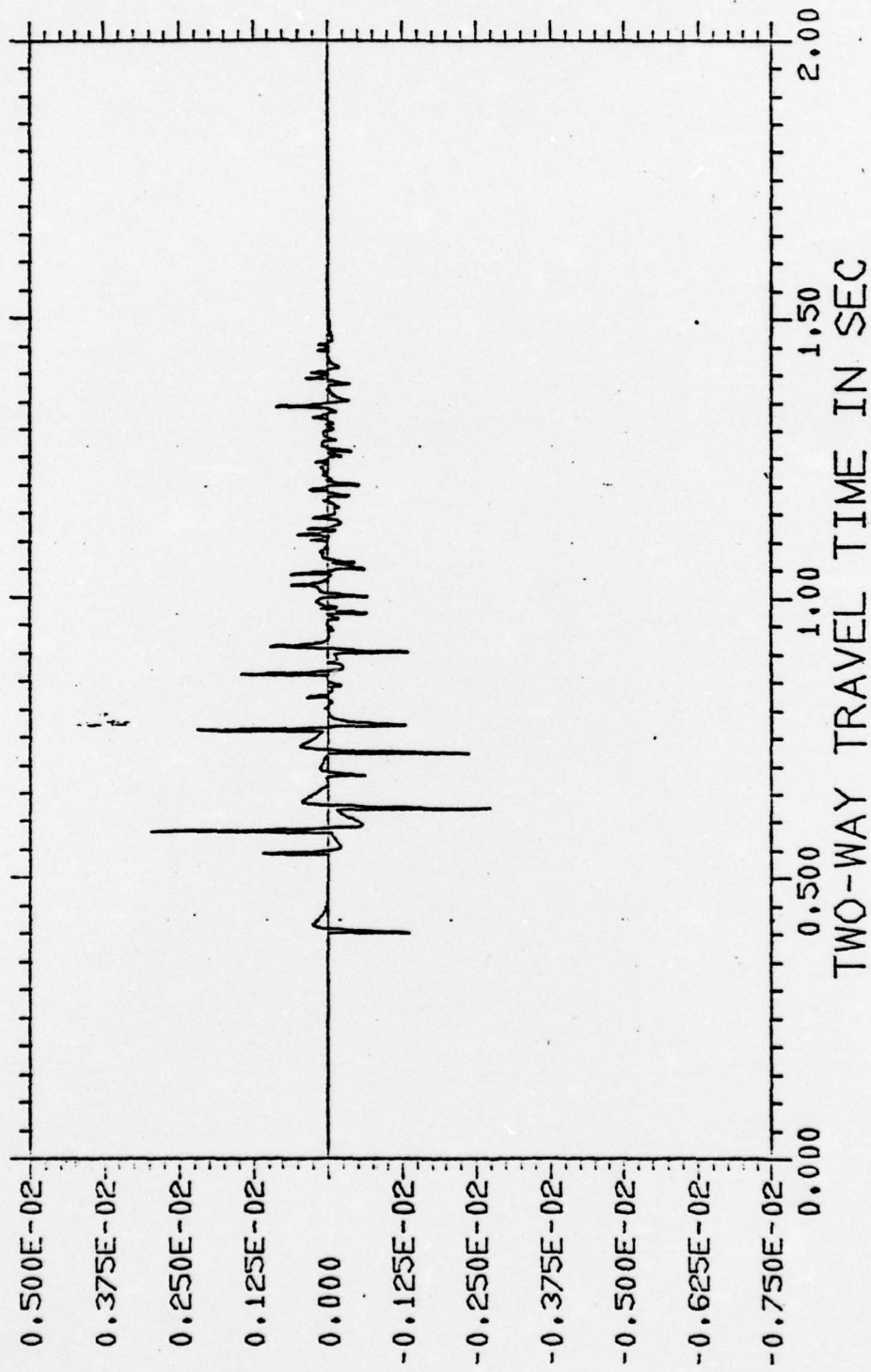


Figure 28. Complete response without some of the water bottom reverberations.

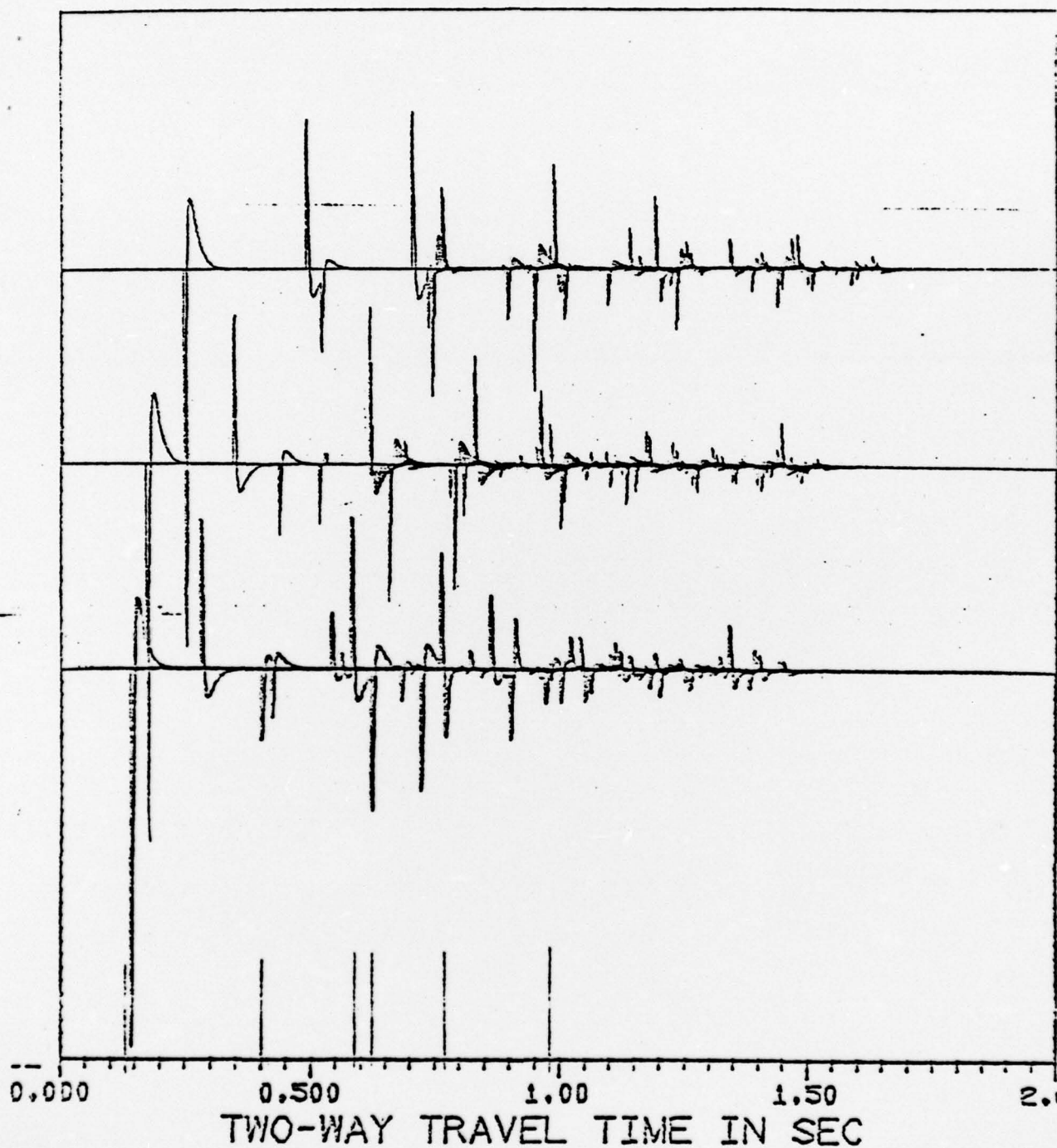


Figure 29. Complete response sensed at normal incidence (lower curve), 500 ft. (middle curve), and 1000 ft. from source. The top two curves were obtained by means of our approximate non-normal incidence transformations, described in Section VII.